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METHODS OF ORDERING A VECTOR SPACE

By PAUL CONRAD

[Received October 29, 1956]

INTRODUCTION AND PRELIMINARIES. In this paper *order* will always mean *linear order*. D will always denote a division ring with elements $0,1,a,b,c,\ldots$, and the field of real numbers will be indicated by R. Unless otherwise stated, the field R and R considered as a vector space over any of its subfields will be naturally ordered. Let A be a left vector space over D with elements $\bar{0},\bar{a},\bar{b},\bar{c},\ldots$. The dimension of A will be denoted by $d[A]$. $\bar{D}$ will denote the left vector space determined by D.

A is *ordered* if A is an ordered group and D is an ordered ring such that $0<d\in D$ and $\bar{0}<\bar{a}\in A$ imply $\bar{0}<d\bar{a}$. An ordering of A is determined by the sets of positive elements of A and D. That is, A can be ordered if and only if there exist subsets P_D and P_A of D and A respectively such that P_D is a subgroup of index 2 in the multiplicative group D^* of D that is closed with respect to addition, and P_A satisfies :

(i) If $\bar{0}\neq\bar{a}\in A\smallsetminus P_A$, then $-\bar{a}\in P_A$. $\bar{0}\notin P_A$.

(ii) P_A is closed with respect to addition.

(iii) P_A is closed with respect to multiplication by elements of P_D.

For, one can define $a>b$ if $a-b\in P_D$, and $\bar{a}>\bar{b}$ if $\bar{a}-\bar{b}\in P_A$.

A can be ordered if and only if D can be ordered. For choose and order a basis $B=\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_\alpha,\ldots$ for A. Then each non-zero element $\bar{a}$ in A has a unique representation $\bar{a}=a_1\bar{b}_{\alpha_1}+a_2\bar{b}_{\alpha_2}+\cdots+a_n\bar{b}_{\alpha_n}$, where $0\neq a_i\in D$ and $\bar{b}_{\alpha_i}\in B$ for $i=1,2,\ldots,n$, and $\alpha_1<\alpha_2<\cdots<\alpha_n$. Define $\bar{a}$ positive if $a_n>0$. It is easy to show that (i), (ii) and (iii) are satisfied. We show (in the proof of Corollary 3.1) that this is

the only method of ordering A if and only if $d[A]=1$ or $d[A]$ is finite and D is isomorphic to R.

Suppose that A is ordered by the above method. Let C and C' be convex subspaces of A so that C covers C'. That is, $\bar{0}<\bar{a}<\bar{c}\in C(\bar{c}\in C')$ implies that $\bar{a}\in C(\bar{a}\in C')$, $C\supset C'$, and no convex subspace of A lies between C and C', Then C/C' is called a *component* of A, and $d[C/C']=1$. In § 1 the methods of ordering A so that its components have dimension 1 are characterized. It is shown that all orderings of A have this property if and only if $d[A]=1$ or D is isomorphic to R. We also investigate when two orderings of this type are semi-automorphic. In § 3 methods of ordering A so that all of its components have rank greater than 1 are investigated. In particular, we prove that if A can be ordered, $d[A]>1$, and $D\not\cong R$, then A can be ordered so that all of its components have the same dimension, and this dimension is greater than 1.

A non-singular linear transformation (of a vector space) will be called an *isomorphism*. In § 1 it is shown that if A is ordered, then A is order isomorphic to a subspace of the large direct sum of ordered spaces each of which has no proper convex subspaces. In § 3 and § 4 we investigate methods of ordering A so that it has no proper convex subspaces. Finally in § 5 it is shown that A can be represented as a non-abelian ordered group H, and that the order preserving semi-automorphisms of A are equivalent to a group of order preserving isomorphisms of H. Most of this paper was at one time written in terms of H.

The main tool used throughout is valuation theory. We have a valuation of A ([4] or Gravett [7]) and a valuation of D [5]. The concepts and results of [4] are used, but the terminology that originated in [4] is defined again here. Schilling [9] is the best source for the basic valuation theory that is used, and Baer [2] is the best source for the vector space theory that is needed, especially the concepts of the homothetic group, semi-linear-transformations, and the fundamental theorems of projective geometry. The short section in Schilling ([9], pp. 5-7) covers the concepts of Archimedean

and non-Archimedean ordering of groups, and Van der Waerden ([10], Chapter IX), covers these concepts for fields. In particular, we use the well-known theorems that if D admits an Archimedean order, then D is order isomorphic to a sub-field of R, and if S is an Archimedean ordered group, then S is order isomorphic to a subgroup of $(R, +)$.

Quite a few of the results in this paper are valid for partially ordered vector spaces. This is a consequence of the fact that the embedding theorem in [4] applies to partially ordered vector spaces.

1. Orderings of A with components of dimension one. Let Γ be an ordered set of elements $\alpha, \beta, \gamma, \ldots$ such that to each γ in Γ there is assigned a vector space $B(\gamma)$ over D. The Γ-*sum* of the $B(\gamma)$ is the space $V = V(\Gamma, B(\gamma))$ of all vectors $\bar{b} = (\ldots, b_\gamma, \ldots)$ with $b_\gamma \in B(\gamma)$ and $b_\gamma = 0$ for all but an inversely well-ordered subset of Γ. V is a subspace of the large direct sum of the $B(\gamma)$ and contains the small direct sum of the $B(\gamma)$. If the $B(\gamma)$ are ordered, then V has a *natural order*: define $\bar{b} \in V$ positive if the non-zero component b_γ with greatest γ is positive. A subspace U of V is called a Γ-*subspace* if for every $\gamma \in \Gamma$ and every $b \in \Gamma\, B(\gamma)$, there exists an element $\bar{a}$ in U such that $a_\gamma = b$ and $a_\alpha = 0$ for all $\alpha > \gamma$. Clearly the small direct sum Δ of the $B(\gamma)$ is a Γ-subspace of V. If U is a Γ-subspace of V, then U does not necessarily contain Δ, but there exists an automorphism π of V such that $U_\pi \supseteq \Delta$. However, no use will be made of this property.

In [4] the term c-subspace is used instead of Γ-subspace. For in [4] the Γ is essentially fixed, but in this paper Γ will be a member of a set T which we now define. Let S be the set of all ordered pairs of subspaces $A^\gamma \supset A_\gamma$ of A such that $d[A^\gamma/A_\gamma] = 1$. S can be thought of as a set of elements α, β, $\gamma, \ldots$ or as a set of ordered pairs (A^γ, A_γ), whichever is more convenient. Define $\alpha < \beta$ if $A^\alpha \subseteq A_\beta$. This partially orders S, and by Zorn's lemma, every chain in S is contained in a maximal chain of S. Let

$$T = [\Gamma : \Gamma \text{ is a maximal chain of } S].$$

Each $\Gamma \in T$ determines a unique Γ-sum $V(\Gamma, A^\gamma/A_\gamma)$. A vector space B over D is a Γ-*extension* of A if $B \supseteq A$ and for each $\gamma \in \Gamma$ there exists a pair of subspaces $B^\gamma \supset B_\gamma$ of B such that $d[B^\gamma/B_\gamma]=1$, $B^\gamma = B_\gamma + (A \cap B^\gamma)$, and the set of all the pairs B^γ, B_γ form a maximal chain. A is Γ-*closed* if it does not admit any proper Γ-extensions. In [4] it is shown that C is a Γ-subspace of V if and only if V is a Γ-extension of C with respect to the natural maximal chain of V, but we will not use this fact here.

THEOREM 1. *For each $\Gamma \in T$, there exists an isomorphism π of A onto a Γ-subspace of $V(\Gamma, A^\gamma/A_\gamma)$. Moreover if $\bar{a} \in A^\alpha \setminus A_\alpha$, then $(\bar{a}\pi)_\alpha = A_\alpha + \bar{a}$ and $(\bar{a}\pi)_\gamma = A_\gamma$ for all $\gamma > \alpha$. If ∂ is any other such isomorphism, then $\pi^{-1}\partial$ is induced by an automorphism of V. $A\pi = V$ if and only if A is Γ-closed.*

PROOF. This is a special case of the Embedding theorem, the Extension theorem for Isomorphisms, and the Corollary to Theorem 4.1 ([4] pp. 11 and 14). It suffices to show that A is a Γ-space ([4] p. 2). Conditions (a) and (b) are obviously satisfied and (d) is an immediate consequence of (c). Thus we need only show :

(c) For every $\bar{a} \neq \bar{0}$ in A there exists a unique $\gamma \in \Gamma$ such that $\bar{a} \in A^\gamma \setminus A_\gamma$. ($\gamma$ is called the *value* of $\bar{a}$ and denoted by $v(\bar{a})$.) Let $\bar{a}$ be a non-zero element in A, and suppose (by way of contradiction) that there does not exist an element $\gamma \in \Gamma$ such that $\bar{a} \in A^\gamma \setminus A_\gamma$. Let $L = \{\gamma \in \Gamma : \bar{a} \notin A^\gamma\}$. If $\alpha \in L$, $\beta \in \Gamma$ and $\beta < \alpha$, then $\beta \in L$. Hence L is a lower segment (possibly void) of Γ. Let B be the subspace consisting of $\bar{0}$ and the join of all the A^γ for $\gamma \in L$. Then B does not contain $\bar{a}$. If $\gamma \in \Gamma \setminus L$, then $B + D\bar{a} \subseteq A_\gamma$. For in this case $\bar{a} \in A^\gamma$, and hence $\bar{a} \in A_\gamma$. Define $A^\beta = B + D\bar{a}$ and $A_\beta = B$. Then for every $\alpha \in L$ and every $\gamma \in \Gamma \setminus L$, $\alpha < B < \gamma$. Thus we have a chain in S which properly contains Γ, a contradiction.

If D is an ordered division ring, then each $\Gamma \in T$ can be used to define orderings of A. For the vector space $\bar{D}$ has two orderings, according as 1 is positive or negative, and each A^γ/A_γ is isomorphic to $\bar{D}$. For a selected set of orderings of the A^γ/A_γ we have a natural

ordering of $V(\Gamma, A^{\gamma}/A_{\gamma})$. Hence we have an ordering of A. Note that in this ordering

$$\bar{0} \neq \bar{a} \in A \text{ is positive if and only if } A_{v(\bar{a})} + \bar{a} \text{ is positive.} \quad (1)$$

We shall refer to an ordering of A obtained by this method as a *T-ordering,* and the method as the *T-method.* In Theorem 3 it is shown that all orderings of real vector spaces are T-orderings, and that real vector spaces are characterized by this property.

Next suppose that A is ordered. Let Σ be the set of all ordered pairs of convex subspaces A^{γ}, A_{γ} of A such that A^{γ} covers A_{γ}. Σ is ordered by inclusion, and its order type is called the *rank* of the ordering. A^{γ}/A_{γ} is called a *component* of A (with respect to the given order). It is well known and easy to show that the natural homomorphism of A^{γ} onto A^{γ}/A_{γ} induces an ordering of A^{γ}/A_{γ}. Hence $V(\Sigma, A^{\gamma}/A_{\gamma})$ has a natural order. Analagous to Theorem 1 we have

THEOREM 2. *If A is ordered, then there exists an order preserving homomorphism π of A onto a Σ-subspace of $V(A^{\gamma}/A_{\gamma})$. Moreover if $\bar{a} \in A^{\alpha} \setminus A_{\alpha}$, then $(\bar{a}\pi) = A_{\alpha} + \bar{a}$ and $(\bar{a}\pi)_{\gamma} = A_{\gamma}$ for all $\gamma > \alpha$. If ∂ is any other such isomorphism, then $\pi^{-1}\partial$ is induced by an order preserving automorphism of V. $A\pi = V$ if and only if A is Σ-closed.*

For, as in the proof of Theorem 1, it suffices to show that (c) is satisfied. Let $U(\bar{a})$ be the intersection of all convex subspaces of A that contain $\bar{a}$, and let $L(\bar{a})$ be the join of all convex subspaces that do not contain $\bar{a}$. Then clearly $U(\bar{a})$ covers $L(\bar{a})$, and (c) is satisfied. The order of A satisfies equation (1). Hence π obviously preserves order.

COROLLARY 2.1. *If A is ordered in such a way that all of its components have dimension one, then $\Sigma \in T$.*

PROOF. Clearly Σ is a chain in S. Suppose that E is not maximal. Then a pair of subspaces $A^{\beta} \supset A_{\beta}$ such that $d\,[A^{\beta}/A_{\beta}] = 1$ can be adjoined to Σ, and the result is still a chain. Consider $\bar{a} \in A^{\beta} \setminus A_{\beta}$. By property (c), there exists a $\gamma \in \Sigma$ such that $\bar{a} \in A^{\gamma} \setminus A_{\gamma}$. But since we still have a chain, either $A^{\gamma} \subseteq A_{\beta}$ or $A^{\beta} \subseteq A_{\gamma}$, a contradiction.

Thus the set of all orderings of A for which the components have dimension one is the set of all T-orderings of A. T supplies the convex subspaces. For if $\Gamma \in T$, and we order A by the T-method, then Γ is the set of all pairs of convex subspaces A^γ, A_γ such that A^γ covers A_γ. Also $\bigcap_{\gamma\in\Gamma} A_\gamma = \bar{0}$ and $\bigcup_{\gamma\in\Gamma} A^\gamma = A$. Next let C be a proper convex subspace of A, $L = \{\gamma \in \Gamma : A^\gamma \subseteq C\}$ and $U = \{\gamma \in \Gamma$: $C \subseteq A_\gamma\}$. Then L, U is a cut in Γ. That is, $\Gamma = L \cup U$, L and U are non-void, and $\alpha \in L$, $\beta \in U$ imply $\alpha < \beta$. Moreover

$$\bigcup_{\gamma\in L} A^\gamma = C = \bigcap_{\gamma\in U} A_\gamma. \tag{2}$$

For $\bigcup_{\gamma\in L} A^\gamma$ and $\bigcup_{\gamma\in U} A_\gamma$ are convex subspaces and clearly $\bigcup_{\gamma\in L} A_\gamma \subseteq C \subseteq \bigcap_{\gamma\in U} A_\gamma$. If $\bar{a} \in C \setminus \bigcup_{\gamma\in L} A^\gamma$, then $\bigcup_{\gamma\in L} A^\gamma \subset A^{v(\bar{a})} \subseteq C$. But this means that $v(\bar{a}) \in L \setminus L$. If $\bar{a} \in \bigcap_{\gamma\in U} A_\gamma \setminus C$, then $C \subseteq A_{v(\bar{a})} \subset A^{v(\bar{a})} \subseteq \bigcap_{\gamma\in U} A_\gamma$. But this means that $v(\bar{a}) \in U \setminus U$.

Conversely, if L, U is a cut in Γ, then equation (2) defines a proper convex subspace C. Thus we have a 1-1 correspondence between the proper convex subspaces of A and the cuts in Γ.

For the rest of this section assume that A can be ordered. The field R of real numbers can be ordered in one and only one way. The following theorem shows that the real vector spaces are characterized by the property that the only method of ordering them is the T-method.

THEOREM 3. *For all possible orderings of A, the components have dimension one if and only if $d[A] = 1$ or D is isomorphic to R.*

PROOF. If $d[A] = 1$, then $A \cong \bar{D}$. Hence for any ordering A has exactly one component, which, of course, has dimension one. Next suppose that $D \cong R$ and that A is ordered. Assume (by way of contradiction) that there exists a component of A with dimension greater than one. Then without loss of generality A is an ordered real vector space, $d[A] > 1$, and A has no proper convex subspaces. Thus there exist a pair of positive elements $\bar{a}, \bar{b} \in A$ such that

$R\bar{a} \oplus R\bar{b}$ is a subspace of A of dimension 2. Since $R\bar{a} \oplus R\bar{b}$ has no proper convex subspaces, there must exist elements $d, e \in R$ such that $d\bar{a} > \bar{b}$ and $e\bar{b} > \bar{a}$. Otherwise $R\bar{a}$ or $R\bar{b}$ is convex. Since R admits only the natural ordering, there exist positive integers m and n such that $m > d$ and $n > e$. Thus $m\bar{a} > \bar{b}$ and $n\bar{b} > \bar{a}$. But this means that the additive group $R \oplus R$ admits an Archimedean ordering. Hence the group $(R, +)$ has a proper Archimedean extension, a contradiction.

Conversely assume that $d[A] > 1$ and $D \ncong R$. Then by Corrollary 9.1, A can be ordered so that all of its components have dimension greater than one.

Two orderings of A are called *automorphic* if there exists an automorphism π of A such that $P\pi = P'$, where P and P' are the sets of positive vectors for the two orders.

COROLLARY 3.1. *If $d[A] > 1$, then the following are equivalent.*

(a) *Each pair of orderings are automorphic.*

(b) *Each pair of orderings of A with respect to the same ordering of D are automorphic.*

(c) *$d[A]$ is finite and D is isomorphic to R.*

PROOF. Clearly (a) implies (b). Suppose that A satisfies (b). For any of the T-orderings of A, the components have dimension one. Order isomorphisms preserve this property. Hence for all orderings of A, the components have dimension one. Thus by Theorem 3, $D \cong R$. Let $B = \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_\alpha, \ldots$ be a well ordered basis for A. Then each $\bar{0} \neq \bar{a}\ A$ has a unique representation $\bar{a} = a_1 \bar{b}_{\alpha_1} + a_2\bar{b}_{\alpha_2} + \ldots + a_n \bar{b}_{\alpha_n}$, where $0 \neq a_i \in D$ and $\bar{b}_{\alpha_i} \in B$ for $i = 1, 2, \ldots, n$, and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Choose an ordering for D and define $\bar{a}$ positive if $a_n > 0$. This orders A so that its rank is well ordered. Next define $\bar{a}$ positive if $a_1 > 0$. This orders A so that its rank is inversely well ordered. But this pair of orderings are automorphic and such an automorphism preserves rank. Thus the rank is finite. Hence B is finite.

Finally suppose that A satisfies (c). Let $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n$ be a basis for A, and define $\bar{a} = a_1\bar{b}_1 + \ldots + a_n\bar{b}_n$ positive if the non-zero a_i with the largest subscript is positive.

Next consider any other ordering of A. Let Σ be the set of all pairs of convex subspaces A^∂, A_∂ such that A^∂ covers A_∂. For each $\partial \in \Sigma$, pick a positive element $\bar{a}(\partial) \in A^\partial \setminus A_\partial$. It is easy to show that the $\bar{a}(\partial)$ are independent, and since (by Theorem 3) the A^∂/A_∂ have dimension one, the $\bar{a}(\partial)$ form a basis for A. Thus we have $\bar{a}(1) < \bar{a}(2) < \ldots < \bar{a}(n)$, and the mapping of $\bar{b}_i$ onto $\bar{a}(i)$ for $i = 1, \ldots, n$ defines an automorphism of A that preserves the two orderings. We have also shown

COROLLARY 3.2. *If $d[A]$ is finite, then any two T-orderings of A with respect to the same ordering of D are automorphic.*

If $d[A] = n$, then without loss of generality $A = \{(a_1, \ldots, a_n) : a_i \in D\}$. As usual, define $(a_1, \ldots, a_n)$ positive if the non-zero component a_i with greatest subscript is positive. Let Q be the group of all $n \times n$ non-singular matrices over D, and U be the subgroup of Q consisting of those triangular matrices with positive elements on the main diagonal and zeros above the diagonal. Then Q is (isomorphic to) the group of automorphisms of A and U is (isomorphic to) the group of order preserving automorphisms of A. Thus we have a 1-1 mapping of the set of all T-orderings of A which are determined by a fixed ordering of D and the group Q/U. If D admits only one ordering (for example, rationals, algebraic extensions of the rationals, real closed fields), then Q/U represents all the T-orderings of A, and if D is isomorphic to R, then Q/U represents all the orderings of A. The preceding results can be generalized by making use of the concept of a semi-isomorphism (Baer [2], p. 42). Let B and C be vector spaces over the division rings G and H respectively. Then *a semi-isomorphism* of B onto C is a pair (α, β), where α is a ring isomorphism of G onto H and β is a group isomorphism of B onto C such that $(g\bar{b})\beta = (g\alpha)(\bar{b}\beta)$ for all $g \in G$ and all $\bar{b} \in B$. If B and C are ordered and α and β are order preserving, then we say that the *orderings are semi-isomorphic.*

THEOREM 4. *Suppose that D has two orderings that are automorphic and $\Gamma', \Gamma'' \in T$. Then a choice of orderings for the components of Γ' and Γ'' determines two T-orderings of A. These two T-orderings are semi-automorphic if Γ' and Γ'' have the same order type and A is Γ'-closed.*

PROOF. By Theorem 1, $A \cong V(\Gamma', A^{\gamma}/A_{\gamma})$. Let $V(\Gamma')$ be the vector space obtained from $V(\Gamma', A^{\gamma}/A_{\gamma})$ by replacing the A^{γ}/A_{γ} by $\bar{D}$. Then clearly $V(\Gamma') \cong V(\Gamma', A^{\gamma}/A_{\gamma})$. Thus the ordering of A obtained from the 1st ordering of D and Γ' is determined by an isomorphism π' of A onto $V(\Gamma')$. Similarly the ordering of A that is obtained from the 2nd ordering of D and Γ'' is determined by an isomorphism π'' of A onto a Γ''-subspace of $V(\Gamma'')$. Let ∂ be a 1-1 order preserving mapping of Γ'' onto Γ' and let ψ be the automorphism of D that maps the positive elements of the 2nd ordering onto the positive elements of the 1st ordering. For $\bar{a} = (\ldots, a_{\gamma}, \ldots) \in V(\Gamma'')$ define $\bar{a}\pi = (\ldots, [a_{\gamma}\psi]_{\gamma\partial}, \ldots)$. For $d \in D$, $(d\bar{a})\pi = (\ldots, da_{\gamma}, \ldots)\pi = (\ldots, [(da_{\gamma})\psi]_{\gamma\partial}, \ldots) = d\psi(\ldots, [a_{\gamma}\psi]_{\gamma\partial}, \ldots) = d\psi\bar{a}\pi$. Thus (ψ, π) is an order preserving semi-isomorphism of $V(\Gamma'')$ onto $V(\Gamma')$. $A\pi'\pi^{-1} = V(\Gamma'')$. Thus by Theorem 1, A is Γ''-closed, and $A\pi'' = V(\Gamma'')$. Therefore $\pi''\pi\pi'^{-1}$ is the desired semi-automorphism.

From the proof of Theorem 6, it is clear that the hypothesis in Theorem 4, that A is Γ'-closed, can be replaced by the condition that Γ' is well ordered. Thus we have

COROLLARY 4.1. *If $d[A]$ is finite, then any two T-orderings of A that are determined by automorphic orderings of D are semi-automorphic.*

REMARKS. The hypothesis that Γ' and Γ'' have the same order type is necessary. In fact, the conclusion of Theorem 4 implies that there exists a projectivity (Baer [2], p. 40) of A that maps Γ' onto Γ''. The hypothesis that A is Γ'-closed is clearly not necessary, and the theorem may remain valid without it. Theorem 4 without this hypothesis is equivalent to the following: Let B and C be Γ-subspaces of $V = V(\Gamma, \bar{D})$, where B is ordered by the 1st ordering of D and C is ordered by the 2nd ordering of D. For each $\gamma \in \Gamma$, let $\pi_{\gamma} = \pm 1$.

For $(\ldots, a_\gamma, \ldots) \in V$, define $(\ldots, a_\gamma, \ldots)\pi = (\ldots, [a_\gamma \pi_\gamma]_\gamma, \ldots)$. If $B\pi = C$, then does there exist an order preserving semi-automorphism of V that maps B onto C ?

2. Structure of T. In the last section it was shown that the set of all convex subspaces of a T-ordering of A is determined by the $\Gamma \in T$ that is used to define the ordering. In fact, $\Gamma' = \{A_\gamma : (A_\gamma, A^\gamma) \in \Gamma\}$ determines the set of all convex subspaces. Let Q be the set of all subspaces of A except A itself. Then $Q = \{A_\gamma : (A_\gamma, A^\gamma) \in S\}$, and we have a mapping of $\Gamma \in T$ onto the chain $\Gamma' \in Q$. Γ' is not necessarily maximal.

THEOREM 5. *Let $\Gamma \in T$, and let Γ' be the chain in Q that is induced by Γ. Then the following are equivalent.*

(a) *Γ' is maximal in Q.*

(b) *Γ' is closed under intersections.*

(c) *Γ' is well ordered.*

PROOF. Let U be a non-void subset of Γ', $X = \{A_\gamma \in \Gamma' : A_\gamma \subset A_\alpha$ for all $A_\alpha \in U\}$ and $Y = \{A_\gamma \in \Gamma' : A_\gamma \supseteq A_\alpha$ for some $A_\alpha \in U\}$. Then $\Gamma' = X \cup Y$, and $A_\alpha \subseteq \bigcap_{A_\gamma \in U} A_\gamma \subseteq A_\beta$ for all $A_\alpha \in X$ and all $A_\beta \in Y$. Thus clearly (a) implies (b). If (*b*) is satisfied, then $\bigcap_{A_\gamma \in U} A_\gamma = A_\delta \in \Gamma'$, A_δ is covered by A^δ, and $U \cup \{A^\delta\}$ is a chain. Therefore A_δ is the least element in U. Hence (b) implies (c). Finally assume that (c) is satisfied, and assume (by way of contradiction) that there exists a subspace B of A such that $B \neq A$, $B \notin \Gamma'$, but $\Gamma' \cup \{B\}$ is a chain. Then either $B \supset A_\gamma$ for all $\gamma \in \Gamma'$ or there exists a least $\gamma \in \Gamma'$ such that $B \subset A_\gamma$. In either case γ is not maximal, a contradiction.

Let $\Gamma'' = \{B : (B,C) \in \Gamma$ or $(C,B) \in \Gamma$ for some subspace C of $A\}$. Then $\Gamma'' = \Gamma'$, if and only if Γ' is well ordered. Since Γ and Γ' have the same order type, Γ' is maximal if and only if Γ is well ordered. By a similar argument it follows that a maximal chain Γ' in Q is induced by a $\Gamma \in T$ if and only if Γ' is well ordered. Thus for our purposes the set S is preferable to the set Q. On the other hand, the

Next theorem shows that each well ordered $\Gamma \in T$ completely determines A, and at least one such Γ always exists.

We shall denote the cardinal number of a set X by $|X|$. Let $Y = \{|\Gamma| : \Gamma \in T\}$, and let L be the least element in Y.

THEOREM 6. *There exists a well ordered* $\Gamma \in T$. *For each well ordered* $\Gamma \in T$, *there exists an isomorphism of* A *onto* $V(\Gamma, A^{\gamma}/A_{\gamma})$ *and* $|\Gamma| = d[A]$. $d[A]$ *is the greatest element in* Y.

PROOF. Let K be a well ordered set such that $|K| = d[A]$, and let E be the set of all finitely non-zero mappings of K into D. That is, E is the small direct sum of $\overline{D}$ taken $|K|$ times. For every $k \in K$, define

$E^k = \{f \in E : f(x) = 0$ for all $x \in K$ such that $x > k\}$, and

$E_k = \{f \in E : f(x) = 0$ for all $x \in K$ such that $x \geqslant k\}$.

The set Σ of all the pairs E^k, E_k is well ordered and is called *the natural maximal chain of* E. Since $A \cong E$, there exists a well ordered $\Gamma \in T$, the image of Σ.

Next suppose that Γ is a well ordered member of T. Then $V(\Gamma, A^{\gamma}/A_{\gamma})$ is the small direct sum of the A^{γ}/A_{γ}, and it has no proper Γ-subspaces ([4], p. 8). Thus by Theorem 1, there exists an isomorphism of A onto V. Therefore $d[A] = d[V] = |\Gamma|$.

If Γ is any element in T, then by Theorem 1, $A \cong C$, where C is a Γ-subspace of $V(\Gamma, A^{\gamma}/A_{\gamma})$. For each $\gamma \in \Gamma$, pick an element $\bar{c}(\gamma) \in C$ such that $v(\bar{c}(\gamma)) = \gamma$. Then the set of all the $\bar{c}(\gamma)$ is an independent set. Therefore $d[A] = d[C] \geqslant |\Gamma|$.

COROLLARY 6.1. $L = d[A]$ *if and only if there exists a well ordered* $\Gamma \in T$ *such that* $|\Gamma| = L$. L *is finite if and only if* $d[A]$ *is finite. If* $d[A] \leqslant \mathcal{N}_0$, *then* $L = d[A]$.

Suppose that $d[A] = \mathcal{N}_{\mu}$, and let $Z(\mathcal{N}_{\mu})$ be the set of all ordinal numbers which can represent $\mathcal{N}_{\mu}$. Then $|Z(\mathcal{N}_{\mu})| = \mathcal{N}_{\mu+1}$ (see for example, Kamke [8], p. 125). *Thus for a given ordering of* D *we have* $\mathcal{N}_{\mu+1}$ *non-automorphic* T*-orderings of* A *with well ordered rank.*

THEOREM 7. *Suppose that* $d[A]$ *is infinite.* (i) L *is the smallest infinite cardinal such that* $|D|^L \geqslant d[A]$. *In particular, if* $|D| \geqslant d[A]$, *then* $L = \aleph_0$. (ii) *If* $L = d[A]$, *then* $d[A]$ *is a limit number.* (iii) *Suppose that* $|D|^X = d[A]$ *has an infinite solution. Then* $|D|^L = d[A]$, *and there exists an inversely well ordered* $\Gamma \in T$ *such that* $|\Gamma| = L$. *For every such* Γ, *there exists an isomorphism of* A *onto* $V(\Gamma, A^\gamma/A_\gamma)$.

PROOF. (i) Let Γ be a member of T for which $|\Gamma| = L$. Then since A is isomorphic to a subgroup of $V = V(\Gamma, A^\gamma/A_\gamma)$, $d[A] \leqslant d[V] \leqslant |V| \leqslant |D|^L$. Let Z be any infinite cardinal such that $d[A] \leqslant |D|^Z$. Let Γ be an inversely well ordered set for which $|\Gamma| = Z$. For each $\gamma \in \Gamma$, let $B(\gamma) = \overline{D}$. Then $V(\Gamma) = V(\Gamma, B(\gamma))$ is the large direct sum of $\overline{D}$ taken Z times. Hence $d[V(\Gamma)] = |D|^Z$ (Baer [2], p. 19). Since $|D|^Z \geqslant d[A]$, there exists an isomorphism π of A onto $V(\Gamma)$. $V(\Gamma)$ has a natural maximmal chain (see the proof of Theorem 6). Hence by using π^{-1} we get a maximal chain $\Gamma \in T$ such that $|\Gamma| \leqslant Z$. Therefore $L \leqslant Z$.

(ii) Suppose that $L = d[A]$ and L is not a limit number. Then there exists an infinite immediate predecessor K of L. $|D|^K \geqslant 2^K \geqslant L = d[A]$. By (i), $L \leqslant K < L$, a contradiction.

(iii) If $|D|^X = d[A]$ has an infinite solution, then by (i), $|D|^L = d[A]$. By the argument used in (i), there exists an inversely well ordered $\Gamma \in T$ such that $|\Gamma| \leqslant L$. Thus $|\Gamma| = L$. The last statement is a consequence of $V(\Gamma, A^\gamma/A_\gamma) \cong V(\Gamma)$, and $d[V(\Gamma)] = |D|^L = d[A]$.

COROLLARY 7.1. *If* $|D| \geqslant d[A]$, *then* $L = d[A]$ *if and only if* $d[A] \leqslant \aleph_0$.

3. Methods of ordering A so that all components have dimension greater than one.

THEOREM 8. *Suppose that* $d[A] > 1$, D *is not isomorphic to* R *and* D *has an Archimedean order. Then* A *can be ordered so that all of its components have the same dimension and this dimension is greater than one.*

PROOF. Without loss of generality D is a proper subfield of R. Hence R is a vector space over D (notation $\bar{R}$) that has no proper convex subspaces and has infinite dimension. If $d[A] \leqslant d[\bar{R}]$, then A is isomorphic to a subspace of $\bar{R}$. Hence A can be ordered so that it has no proper convex subspaces. If $d[A] > d[\bar{R}]$, then $d[A] = (d[A])\ (d[\bar{R}])$. Thus A is the small direct sum of $d[A]$ subspaces S_α, each of which is isomorphic to $\bar{R}$. Order this set of subspaces S_α, and define $\bar{a} \in A$ positive if its non-zero component from the greatest S_α is positive. This definition orders A so that each of its components is order isomorphic to $\bar{R}$.

We next develop a method for ordering A so that it has no proper convex subspaces *provided that D has a non-Archimedean order.* Represent A as the set of all vectors $\bar{a} = (\ldots, a_\gamma, \ldots)$, where $a_\gamma \in D$ and all but a finite number of the a_γ are zero. Denote the set of all subscripts by Γ. This amounts to choosing a basis for A. Let v be the order valuation of D ([5], p. 325). Then v is a homomorphism of the multiplicative group D^* of D onto an additive ordered group Π, called the *value group*. Denote the identity of Π by θ. Then $D^\theta = \{d \in D : v(d) \leqslant \theta\}$ is the *valuation ring* of D, and D^θ/D_θ is the residue class ring (where $D_\theta = \{d \in D : v(d) < \theta\}$). The natural homomorphism of D^θ onto D^θ/D_θ induces an ordering of D^θ/D_θ, and D^θ/D_θ is order isomorphic to a subfield R_θ of R. Thus we have a ring homomorphism ∂ of D^θ onto R_θ, with kernel D_θ, that preserves order. Let $A^\theta = \{(\ldots, a_\gamma, \ldots) \in A : a_\gamma \in D^\theta\}$. Then A^θ is an abelian group that admits scalar multiplication by elements in D^θ. For $\bar{a} = (\ldots, a_\gamma, \ldots)$ in A^θ we define $\bar{a}\pi = (\ldots, a_\gamma\partial, \ldots)$. Then π is a homomorphism of A^θ onto the vector space $V(R_\theta)$, where $V(R_\theta)$ is the small direct sum of R_θ taken $|\Gamma|$ times. The kernel of π is $A_\theta = \{(\ldots, a_\gamma, \ldots) \in A : a_\gamma \in D_\theta\}$. If $d \in D^\theta$ and $(\ldots, a_\gamma, \ldots) \in A^\theta$, then $[d(\ldots, a_\gamma, \ldots)]\pi = (\ldots, da_\gamma, \ldots)\pi = (\ldots, d\partial a_\gamma\partial, \ldots) = d\partial\ (\ldots, a_\gamma, \ldots)\pi$. Next order $V(R_\theta)$ by any method. (Any T-ordering will do, and if $R_\theta \neq R$, then $V(R_\theta)$, or part of it, can be ordered by the method given in the proof of Theorem 8.) Define $\bar{a} \in A^\theta$ positive if $\bar{a}\pi$ is positive in $V(R_\theta)$. Thus we have a *partial* ordering of the group A^θ that satisfies :

(a) if $\bar{a} \in A^\theta \smallsetminus A_\theta$, then $\bar{a}$ or $-\bar{a}$ is positive;

(b) if $\bar{a} \in A^\theta$ is positive and $0 < d \in D^\theta \smallsetminus D_\theta$, then $d\bar{a}$ is positive.

Note that the natural homomorphism of A^θ onto $A^\theta \smallsetminus A_\theta$ induces an ordering of A^θ/A_θ. Now consider a non-zero element $\bar{a} \in A$. Let x be the maximum value of its components a_γ, and let d be a positive element in D such that $v(d) + x = \theta$. Then $d\bar{a} \in A^\theta \smallsetminus A_\theta$.

Define $\bar{a}$ positive if $d\bar{a}$ is positive in A^θ.

This definition is independent of the particular choice of d. For if $0 < c \in D$ and $v(c) + x = \theta$, then $v(c) = v(d)$. Hence $v(cd^{-1}) = v(c) - v(d) = \theta$. Thus $0 < cd^{-1} = b \in D^\theta \smallsetminus D_\theta$. By property (b), $d\bar{a}$ is positive if and only if $bd\bar{a} = c\bar{a}$ is positive.

Let P^θ and P be the sets of positive elements in A^θ and A respectively. (i) Suppose that $\bar{0} \neq \bar{a} \in A \smallsetminus P$. Choose $0 < d \in D$ such that $v(d) + x = \theta$, where x is the maximum value of the components of $\bar{a}$. Then $d\bar{a} \in A^\theta \smallsetminus A_\theta$ and has at least one component with value θ. Thus by (a), $-d\bar{a} \in P^\theta$. But $-d\bar{a} = d(-\bar{a})$ and $v(c) = v(-c)$ for all $c \in D$. Therefore $-\bar{a} \in P$. (ii) Suppose that $\bar{a}\,\bar{b} \in P$. Then $\bar{a} = (\ldots, a_{\alpha_1}, \ldots, a_{\alpha_2}, \ldots, a_{\alpha_m}, \ldots)$, where $v(a_{\alpha_i}) = x$ and $v(a_\gamma) < x$ if $a_\gamma \neq a_{\alpha_i}$ for $i = 1, 2, \ldots, m$. $\bar{b} = (\ldots, b_{\beta_1}, \ldots, b_{\beta_2}, \ldots, b_{\beta_n}, \ldots)$, where $v(b_{\beta_i}) = y$ and $v(b_\gamma) < y$ if $b_\gamma \neq b_{\beta_i}$ for $i = 1, 2, \ldots, n$. Choose positive elements r and s in D such that $v(r) + x = \theta$ and $v(s) + y = \theta$. In particular, if $x = y$, choose $r = s$. Then $r\bar{a}$, $s\bar{b} \in P^\theta$. If $x > y$, then $v(s) > v(r)$, and $r(\bar{a} + \bar{b}) = r\bar{a} + r\bar{b} \equiv r\bar{a}$ modulo A_θ. Therefore $\bar{a} + \bar{b} \in P$. Next suppose that $x = y$. Assume that the maximum value of the components in $\bar{a} + \bar{b}$ is less than x. Then $m = n$, $\alpha_i = \beta_i$ and $a_{\alpha_i} \equiv -b_{\beta_i}$ modulo D_θ for $i = 1, 2, \ldots, n$. But this means that either $r\bar{a}$ or $r\bar{b}$ is negative in A^θ, a contradiction. Therefore the maximum value of the components of $\bar{a} + \bar{b}$ is x. Hence we can use r on $\bar{a} + \bar{b}$. But $r(\bar{a} + \bar{b}) = r\bar{a} + r\bar{b} \in P^\theta$. Thus in either case $\bar{a} + \bar{b} \in P$. (iii) Suppose that $\bar{a} \in P$ and $0 < f \in D$. Then choose $0 < d \in D$ as we did in the proof of (i). Then $d\bar{a} = (df^{-1})f\bar{a}$ and $df^{-1} > 0$. Therefore $f\bar{a} \in P$. Thus this definition orders A. Note that the hypothesis,

that the ordering of D is non-Archimedean, has not been used. (If D has an Archimedean order, then $D \cong R_\theta$ and $A \cong V\ (R_\theta)$.) We only need this hypothesis to show that A has no proper convex subspaces.

Let C be a non-zero convex subspace of A, and consider $\bar{c} \in C \cap P$ and $\bar{a} \in A \cap P$. Then there exists a positive element $d \in D$ such that $v(d) + y > x$, where x (y) is the maximal value of the components of $\bar{a}(\bar{c})$. It follows that $\bar{0} < \bar{a} < d\bar{c} \in C$. Therefore $A = C$, and we have proved

THEOREM 9. *If D has a non-Archimedean order, then A can be ordered so that it has no proper convex subspaces.*

Combining Theorems 8 and 9 we have

COROLLARY 9.1. *If A can be ordered, $d[A] > 1$, and D is not isomorphic to R, then A can be ordered so that all of its components have the same dimension, and this dimension is greater than one.*

We shall refer to an ordering of A obtained by the method used in the proof of Theorem 9 as an *N-ordering* and the method as the *N-method*. Note that if $R_\theta = R$, then the N-orderings of A are determined by the T-orderings of A^θ/A_θ.

Suppose that D is ordered, and let B be the basis for A. Define $A_\theta(B) = \{a_1\bar{b}_{\beta_1} + \ldots + a_n\bar{b}_{\beta_n} : a_i \in D_\theta \text{ and } \bar{b}_{\beta_i} \in B \text{ for } i = 1, \ldots, n\}$ and $A^\theta(B) = \{a_1\bar{b}_{\beta_i} + \ldots + a_n\bar{b}_{\beta_n} : a_i \in D^\theta \text{ and } \bar{b}_{\beta_i} \in B \text{ for } i = 1, \ldots, n\}$. An N-ordering of A depends upon the ordering of D, the choice of B, and the choice of an ordering for $A^\theta(B)/A_\theta(B)$. Note that $A^\theta(B)/A_\theta(B)$ is (isomorphic to) a vector space over a subfield R_θ of the reals and has the same dimension as A.

COROLLARY 9.2. *An ordering of A is an N-ordering if and only if there exists a basis B for A such that the natural homomorphism of $A^\theta(B)$ onto $A^\theta(B)/A_\theta(B)$ induces an ordering of $A^\theta(B)/A_\theta(B)$.*

For, the necessity of this condition was shown in the proof of Theorem 9, and the sufficiency follows by an easy argument.

4. Orderings of A with no proper convex subspaces. By Theorem 2, an ordered vector space is order isomorphic to a Γ-subspace of a Γ-sum of ordered vector spaces each of which has no proper convex subspaces. If the condition $d[A^{\gamma}/A_{\gamma}] = 1$ is omitted from the definition of S (that is, S is the set of all pairs of convex subspaces $A^{\gamma} \supset A_{\gamma}$ of A), then Theorem 1 remains valid. In fact, the same proof will do. Hence if the A^{γ}/A_{γ} can be ordered so that they have no proper convex subspaces, then the natural ordering of $V(\Gamma, A^{\gamma}/A_{\gamma})$ induces an ordering of A for which the A^{γ}/A_{γ} are the components. Thus it is important to investigate the orderings of a vector space with no proper convex subspaces. If A is ordered and $d(A] = 1$, *then* A *is order isomorphic to* $\bar{D}$, where the ordering of $\bar{D}$ is the natural one induced by the ordering of D. By Theorem 9, if D admits a non-Archimedean order, then there exists an ordering of A with no proper convex subspaces. Thus if $|D| > |R|$, then A has at least one such ordering (provided that D can be ordered).

THEOREM 10. *If A is ordered, D has an Archimedean order and A has no proper convex subspaces, then D is order isomorphic to a subfield D' of R and A is order isomorphic to a subspace of the vector space $\bar{R}$ of R over D'.*

PROOF. Without loss of generality $D \subseteq R$. A has no proper convex subgroups. For let C be a non-zero convex subgroup of A, and consider $\bar{0} < \bar{c} \in C$ and $\bar{0} < \bar{a} \in A$. Since A has no proper convex subspaces, there exists a positive element $d \in D$ such that $\bar{0} < \bar{a} < d\bar{c}$. But since $D \leqslant R$, there exists a positive integer n such that $d < n$. Hence $\bar{0} < \bar{a} < d\bar{c} < n\bar{c} \in C$. Thus $C = A$. Therefore there exists an order preserving isomorphism π of the group $(A, +)$ into the group $(R, +)$. By the standard arguments it follows that $(r\bar{a})\pi = r(\bar{a}\pi)$ for all $\bar{a} \in A$ and for every rational number r. By the usual continuity arguments it follows that $(d\bar{a})\pi = d(\bar{a}\pi)$ for all $\bar{a} \in A$ and all $d \in D$.

The following corollary is an immediate consequence of Theorems 9 and 10.

COROLLARY 10.1. *A can be ordered so that it has no proper convex subspaces if and only if one of the follwing is satisfied :*

(a) *D admits a non-Archimedean order;*

(b) *D is (isomorphic to) a subfield of R and $d[A] \leqslant d[\bar{R}]$.*

Suppose that A can be ordered and let F be the prime field of D. Then without loss of generality F is the field of rationals. For each basis B of A we define $F(B) = \{a_1 \bar{b}_{\beta_1} + \dots + a_n \bar{b}_{\beta n} : a_i \in F$ and $\bar{b}_{\beta_i} \in B$ for $i = 1, \dots, n\}$. $F(B)$ is a vector space over F and $d[F(B)] = d[A]$.

THEOREM 11. *If A is ordered and there exists a basis B of positive elements for A such that $F(B)$ has no proper convex subspaces, then*

(*) $\sum_{i=1}^{n} a_i \bar{b}_{\beta_i}$ *is positive provided one of the components a_i is positive and has greater value than the other components.*

PROOF. (*) is obvious for $n = 1$. Assume that (*) is true for $n - 1$. Let $\bar{a} = \sum_{n=1}^{n} a_i \bar{b}_{\beta_i}$, and suppose that $a_1 > 0$ and $v(a_1) > v(a_i)$ for $i = 2, \dots, n$. If $a_2 > 0$, then $\bar{a} = a_2\bar{b}_{\beta_2} + \bar{b}$, and $\bar{b} = \bar{a} - a_2\bar{b}_{\beta_2}$ is positive by our induction assumption. Thus $\bar{a} > \bar{0}$. Suppose that $a_2 = -f$, where $f > 0$.

Case I. $\bar{b}_{\beta_1} > \bar{b}_{\beta_2}$. Then $f(\bar{b}_{\beta_1} - \bar{b}_{\beta_2}) > \bar{0}$ and $v(a_1) > v(a_2) = v(f)$. Thus $a_1 - f = d > 0$ and $v(d) = v(a_1)$. Hence $\bar{a} = f(\bar{b}_{\beta_1} - \bar{b}_{\beta_2}) + \left(d\bar{b}_{\beta_1} + \sum_{i=3}^{n} a_i \bar{b}_{\beta_2} \right)$. By induction the last term is positive. Thus $\bar{a} > \bar{0}$.

Case II. $\bar{b}_{\beta_1} < \bar{b}_{\beta_2}$. There exists an element $a \in F$ such that $a\bar{b}_{\beta_1} > \bar{b}_{\beta_2}$. For otherwise $\{\bar{b} \in F(B) : -c\bar{b}_{\beta_2} < \bar{b} < c\bar{b}_{\beta_2}$ for all $0 < c \in F\}$ is a proper convex subspace of $F(B)$. Consider any $0 < e \in D$ such that $v(e) > \theta$. Then $e > a$, hence $e\bar{b}_{\beta_1} > a\bar{b}_{\beta_1} > \bar{b}_{\beta_2}$, and $fe\bar{b}_{\beta_1} - f\bar{b}_{\beta_2} > \bar{0}$. Now let e be the solution of $fx = (1/2)a_1$. Then $v(e) > \theta$ and $\bar{a} = (fe\bar{b}_{\beta_1} - f\bar{b}_{\beta_2}) + \left((1/2)a_1 \bar{b}_{\beta_1} + \sum_{i=3}^{n} a_i \bar{b}_{\beta_i}\right)$.

Once again, the last term is positive. Therefore $\bar{a} > \bar{0}$.

Remark. The hypothesis of Theorem 11 implies that $d[A] \leq |R|$ but it does not imply that A has no proper convex subspaces (see below Method II). Also there are orderings of A with no proper convex subspaces that do not satisfy (*). But this theorem does restrict the possible methods of ordering the direct sum of a set of ordered vector spaces of dimension one. For if $F(B)$ is ordered as a subspace of $\bar{R}$, then (*) must be used as part of the definition.

We conclude this section by giving two methods of ordering a vector space of dimension two so that it has no proper convex subspaces. In general, these orderings are non-automorphic and neither is an N-ordering. For the remainder of this section assume that *D has a non-Archimedean order and that* $d[A] = 2$.

Method I. Let L, U be a cut in the value group Π of D. That is, L and U are non-void subsets of Π such that $\Pi = L \cup U$ and $\alpha \in L, \beta \in U$ imply $\alpha < \beta$. Define $(a, b) \in A$ positive if

(a) $ab = 0$ and $a + b > 0$, or

(b) $ab \neq 0$, $v(a^{-1}b) \in U$ and $b > 0$, or

(c) $ab \neq 0$, $v(a^{-1}b) \in L$ and $a > 0$.

By straightforward grinding it follows that the set P of all positive elements in A satisfies conditions (i), (ii) and (iii). Let C be a non-zero convex subspace of A, and consider $(a, b) \in A \cap P$ and $(c, d) \in C \cap P$. Then there exists a positive element $f \in D$ such that $\bar{0} < (a, b) < (fc, fd)$. Hence $C = A$. For if $c > 0$ and $d > 0$, then for large f, $fc - a > 0$ and $fd - b > 0$. If $c > 0$, and $d = 0$, then for large f, $fc - a > 0$ and if $b \neq 0$, $v((fc - a)^{-1}(-b)) = -v(fc - a) + v(b) = -(fc - a) + v(b) \in L$. If $c > 0$ and $d < 0$, then $v(c^{-1}d) \in L$. For large f, $fc - a > 0$ and $v((fc - a)^{-1}(fd - b)) = -v(fc - a) + v(fd - b) = -v(fc) + v(fd) = -v(c) - v(f) + v(f) + v(d) = v(c^{-1}d) \in L$. Similarly for the other two cases, $(fc - a, fd - b) > \bar{0}$ for large f.

Method II. Let D' be any ordered extension of D. For example, (if D is a field) let D' be the simple transcendental extension of D by x, and define $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ positive if $a_n > 0$. This extends the ordering of D to an ordering of the polynomial

ring $D[x]$, and this order can be extended to an ordering of D' in one and only one way. Pick $0 < q \in D' \smallsetminus D$ such that $v(d) = \theta$. For example, let $q = -x^{-1} + 1$. Define $(a, b) \in A$ positive if

(1) $v(a) > v(b)$ and $a > 0$, or

(2) $v(a) < v(b)$ and $b < 0$, or

(3) $v(a) = v(b)$ and $a + bq > 0$.

Once again, by a straightforward argument it follows that (i), (ii) and (iii) are satisfied. Let C be a non-zero convex subspace of A and consider $\bar{0} < (a, b) \in A$ and $\bar{0} < (c, d) \in C$. If $v(c) > v(d)$, then $c > 0$. Hence for large f, $fc - a > 0$ and $v(cf - a) > v\ (fd - b)$. Similarly if $v(c) < v(d)$, then $(a, b) \in C$. Finally suppose that $v(d) = v(c)$. Then $c + dq > 0$. For large f, $v(fc - a) = v(fd - b)$ and $fc - a + (fd - b)q = f(c + dq) - a - bq > 0$. Therefore $C = A$.

Remarks. If we use the D' and the q suggested above, then $(1, -1)$ is the least element of the form $(a, -1)$ that is positive. If D' is an a-extension of D ([5], p. 324), then for any choice of q such that $0 < q \in D' \smallsetminus D$ and $v(q) = \theta$, there will be no least positive element of the form $(a, -1)$. In fact it can be shown that this property characterizes the a-extensions of D. If we choose $0 < q \in D$ such that $v(q) = \theta$ and add the condition, (4) $v(a) = v(b)$, $a + bq = 0$ and $a > 0$, to our definition of order, then this is a T-ordering of A. The only proper convex subspace is $D(q, -1)$. In particular, if $q \in D \smallsetminus F$, then $F(B) = \{(x, y) : x, y \in F\}$ has no proper convex subspaces.

5. Representation of an ordered vector space as an ordered group.

In this section let A be an ordered *right* rector space over D, and let o-isomoprhism denote order preserving isomorphism, o-group denote ordered group, o-extension denote an ordered extension that induces the given ordering, etc. Let P_D be the multiplicative group of positive elements in D, and let

$$H = \{(a, \bar{a}) : a \in P_D \text{ and } \bar{a} \in A].$$

For $(a, \bar{a})$ and $(b, \bar{b})$ in H, define $(a, \bar{a})\ (b, \bar{b}) = (ab, \bar{a}b + \bar{b})$, and $(a, \bar{a})$ positive if $a > 1$ or $a = 1$ and $\bar{a} > \bar{0}$.

I. H is isomorphic to the group of all mappings of A of the form $x' = xa + \bar{a}$, where $a \in P_D$ and $\bar{a} \in A$. Thus H is (isomorphic to) a subgroup of the homothetic group (Baer [2], p. 53) which consists of all the mappings of A into itself of the form $x' = xa + \bar{a}$, where $a \in D^*$ and $\bar{a} \in A$.

II. H is a normal splitting o-extension of $N = \{(1, \bar{a}) : \bar{a} \in A\}$ by $E = \{(a, \bar{0}) : 0 < a \in D\}$ ([6], p. 518). N is o-isomorphic to the group $(A, +)$ and E is o-isomorphic to the group P_D.

For $(a, \bar{a}) \in H$ and $(1, \bar{b}) \in N$,

$$(a, \bar{a})^{-1}(1, \bar{b})\,(a, \bar{a}) = (a^{-1}, -\bar{a}a^{-1})\,(a, \bar{b}a + \bar{a}) = (1, \bar{b}a).$$

Thus we have

III. Let $C' = \{(1, \bar{c}) : c \in C\}$, where C' is a convex subspace of A. Then C' is a normal convex subgroup of H, and the mapping of C upon C' is a $1-1$ mapping of the set of all convex subspaces of A onto the set of all normal abelian convex subgroups of H.

We will make use of the following groups.

$\mathscr{A}$ = all o-automorphisms of H,

$\mathscr{A}^* = \{\partial \in \mathscr{A} : E\partial = E\}$,

$\mathscr{S}$ = all semi-o-automorphisms of A.

Since N is the greatest abelian convex subgroup of H, we have

IV. $N\partial = N$ for all $\partial \in \mathscr{A}$.

Let $\pi = (\pi', \pi'') \in \mathscr{S}$, and define $(a, \bar{a})\,\pi^* = (a\pi', \bar{a}\pi'')$ for every $(a, \bar{a}) \in H$.

V. The mapping of π upon π^* is an isomorphism of $\mathscr{S}$ onto $\mathscr{A}^*$.

For it is fairly easy to show that this mapping is an isomorphism of $\mathscr{S}$ onto $\mathscr{A}^*$. We omit the details and show only that the mapping is onto. Consider $\pi^* \in \mathscr{A}^*$ and $(a, \bar{a}) \in H$. Then

$$(a, \bar{a})\,\pi^* = (a, \bar{0})\,\pi^*(1, \bar{a})\,\pi^* = (a\pi', \bar{0})\,(1, \bar{a}\pi'') = (a\pi', \bar{a}\pi'').$$

First we show that π' is an o-automorphism of P_D, π'' is an o-automomophism of $(A,+)$ and $(\bar{a}a)\pi'' = \bar{a}\pi''\, a\pi'$ for all $\bar{a} \in A$ and all $a \in P_D$.

$$((ab)\pi', (\bar{a}b + \bar{b})\pi'') = (ab, \bar{a}b + \bar{b})\pi^* = [(a, \bar{a})(b, \bar{b})]\pi^*$$
$$= (a, \bar{a})\pi^*(b, \bar{b})\pi^*. (a\pi', \bar{a}\pi'')(b\pi', \bar{b}\pi'')$$
$$= (a\pi' b\pi', \bar{a}\pi'' b\pi' + \bar{b}\pi'').$$

Thus $(\bar{a}b)\pi' = a\pi' b\pi'$. Hence π' is an endomorphism of P_D, and in particular $1\pi' = 1$. $(\bar{a}b + \bar{b})\pi'' = \bar{a}\pi''\, b\pi' + \bar{b}\pi''$. By letting $b = 1$ we have, $(\bar{a} + \bar{b})\pi'' = \bar{a}\pi'' + \bar{b}\pi''$. Therefore π'' is an endomorphism of $(A, +)$. Thus $(\bar{a}b)\pi'' + \bar{b}\pi'' = (\bar{a}b + \bar{b})\pi'' = \bar{a}\pi''\, b\pi' + \bar{b}\pi''$. Therefore $(\bar{a}b)\pi'' = \bar{a}\pi''\, b\pi'$ for all $\bar{a} \in A$ and all $b \in P_D$. That π' and π'' are automorphisms is an easy consequence of the fact that π^* is an o-automorphism and $N\pi^* = N$.

It remains to be shown that π' can be extended to an o-automorphism ψ of D such that $(\bar{a}a)\pi'' = \bar{a}\pi'' a\psi$ for all $\bar{a} \in A$ and all $a \in D$. (It then follows from V that this extension is unique.) Consider $a, b \in P_D$ and $\bar{0} \neq \bar{a} \in A$.

$$\bar{a}\pi''(a + b)\pi' = [\bar{a}(a + b)]\pi'' = (\bar{a}a + \bar{a}b)\pi'' = (\bar{a}a)\pi'' + (\bar{a}b)\pi''$$
$$= \bar{a}\pi''\, a\pi' + \bar{a}\pi''\, b\pi' = \bar{a}\pi''(a\pi' + b\pi').$$

Therefore $(a + b)\pi' = a\pi' + b\pi'$. For $a \in D$ we define

$$a\psi = \begin{cases} a\pi' & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -[(-a)\pi'] & \text{if } a < 0. \end{cases}$$

Then by considering the various cases, it follows that ψ is the desired o-automorphism. For instance, if $a < 0 < b$ and $a + b = d < 0$, then $-a = b + (-d)$ and $-a$, b and $-d$ are positive. Thus $(-a)\pi' = b\pi' + (-d)\pi'$. Therefore

$$(a + b)\psi = d\psi = -[(-d)\pi'] = -[(-a)\pi'] + b\pi' = a\psi + b\psi.$$

Thus A and $\mathscr{S}$ are determined by H and $\mathscr{A}^*$. Instead of considering H, we could have considered the homothetic group $K = \{(a, \bar{a}) :$

$a \in D^*$ and $\bar{a} \in A\}$. Then the orderings of A correspond to subgroups of K that are normal o-extensions of N by a subgroup of index 2 in D^*. Two orderings of A are semi-automorphic if and only if the corresponding subgroups are o-isomorphic and this o-isomorphism is induced by an automorphism of K. Thus the theorems and methods of ordering that we have derived for A lead to theorems about the structure of K and conversely. We close this paper with an investigation of the structure of $\mathscr{A}$. The result obtained can also be formulated for the group of all automorphisms of K.

Consider $\partial \in \mathscr{A}$. Then $N\partial = N$ and for $(a, \bar{a}) \in H$, $(a, \bar{a})\partial = [(a, \bar{0})(1, \bar{a})]\partial = (a, \bar{0})\partial\ (1, \bar{a})\partial = (a\alpha, a\beta)\ (1, \bar{a}\gamma) = (a\alpha, a\beta + \bar{a}\gamma)$. Thus

(1) $(a, \bar{a})\partial = (a\alpha, a\beta + \bar{a}\gamma)$,

where α is a mapping of P_D into itself, β is a mapping of P_D into A, and γ is an o-automorphism of $(A, +)$. For $(a, \bar{a})$ and $(b, \bar{b})$ in H,

$$\begin{aligned} ((ab)\alpha, (ab)\beta + (\bar{a}b)\gamma + \bar{b}\gamma) &= (ab, \bar{a}b + \bar{b})\partial \\ &= [(a, \bar{a})\ (b, \bar{b})]\partial = (a, \bar{a})\partial\ (b, \bar{b})\partial \\ &= (a\alpha, a\beta + \bar{a}\gamma)\ (b\alpha, b\beta + \bar{b}\gamma) \\ &= (a\alpha\, b\alpha, (a\beta + \bar{a}\gamma)\ b\alpha + b\beta + \bar{b}\gamma) \\ &= (a\alpha\, b\alpha, a\beta\, b\alpha + b\beta + \bar{a}\gamma b\alpha + \bar{b}\gamma). \end{aligned}$$

Therefore

(2) $(ab)\alpha = a\alpha\, b\alpha$;

(3) $(ab)\beta + (\bar{a}b)\gamma = a\beta\, b\alpha + b\beta + \bar{a}\gamma\, b\alpha$.

By (2), α is an endomorphism of P_D. For $\bar{a} = \bar{b} = \bar{0}$, (3) reduces to

(4) $(ab)\beta = a\beta\, b\alpha + b\beta$.

Thus in general, (3) reduces to (4) and

(5) $(\bar{a}b)\gamma = \bar{a}\gamma\, b\alpha$.

If $a\alpha = 1$, then $(a, \bar{0})\partial = (a\alpha, a\beta) = (1, a\beta) \in N$. Thus $(a, \bar{0}) \in N$. Hence $a = 1$, and α is an isomorphism. If $a > 1$, then $(a, \bar{0})\partial = (a\alpha, a\beta)$

is positive and $a\alpha \neq 1$. Thus $a\alpha > 1$. Since ∂ is an automorphism of H, $(x\alpha, x\beta + \bar{y}\gamma) = (x, \bar{y})\partial = (a, \bar{a})$ has a solution $(x, \bar{y})$. Thus $x\alpha = a$ has a solution. Therefore α is an o-automorphism of P_D, and we have

VI. α is an o-automorphism of P_D, γ is an o-automorphism of $(A, +)$, and $(a\bar{a})\gamma = \bar{a}\gamma\, a\alpha$ for all $\bar{a} \in A$ and all $a \in P_D$. Thus (a, γ) is a semi-o-automorphism of the operator group $(A, +)$ with operators P_D, and it is induced by a unique semi-o-automorphism of A (see the proof of V). β is a *crossed homomorphism* of P_D into $(A, +)$ with respect to α.

VII. Conversely suppose that α is an o-automorphism of P_D, γ is an o-automorphism of $(A, +)$, $(\bar{a}a)\gamma = \bar{a}\gamma a\alpha$ for all $\bar{a} \in A$ and all $a \in P_D$, and β is a crossed homomorphism of P_D into $(A, +)$ with respect to α. Then $\partial \in \mathscr{A}$, where ∂ is defined by equation (1).

The proof is easy and we omit the details. Thus to characterize the structure of $\mathscr{A}$, we need to determine the crossed homomorphisms of P_D into $(A, +)$. Let α be an o-automorphism of P_D and $\bar{a} \in A$. For $a \in P_D$ we define : $a\beta = \bar{a}(a\alpha - 1)$. Then

$$a\beta b\alpha + b\beta = \bar{a}\,(a\alpha - 1)\, b\alpha + a\,(b\alpha - 1)$$
$$= \bar{a}\,(a\alpha b\alpha - 1) = \bar{a}\,((ab)\,\alpha - 1) = (ab)\beta.$$

Thus for each $\bar{a} \in A$ there is a crossed homomorphism β of P_D into $(A, +)$ with respect to α. Note that β is 1-1 if $\bar{a} \neq \bar{0}$. In this case β^{-1} is a crossed isomorphism in the sense of Baer [1].

VIII. Suppose that β is a crossed homomorphism of P_D into $(A, +)$ with respect to α. Then $a\beta = \bar{a}\,(a\alpha - 1)$ for all $a \in P_D$ and some fixed $\bar{a} \in A$.

Proof. Let Z be the centre of P_D. Since D is ordered $Z \neq 1$.

Case I. $Z\beta = \bar{0}$. Consider $a \in P_D \smallsetminus Z$, and $1 \neq z \in Z$. Then $z\alpha \neq 1$ and $a\beta = z\beta a\alpha + a\beta = (za)\beta = (az)\beta = a\beta z\alpha + z\beta = a\beta z\alpha$ Therefore $a\beta = \bar{0}$. Hence $a\beta = \bar{0}\ (a\alpha - 1)$ for all $a \in P_D$.

Case II. There exists an element z in Z such that $z\beta \neq 0$. Note $z \neq 1$. $z\beta = z\beta\,(z\alpha - 1)^{-1}\,(z\alpha - 1) = \bar{a}\,(z\alpha - 1)$. For $a \in P_D$,

$$a\beta z\alpha + \bar{a}(z\alpha - 1) = a\beta z\alpha + z\beta = (az)\beta = (za)\beta$$
$$= z\beta a\alpha + a\beta = \bar{a}(z\alpha - 1)\, a\alpha + a\beta.$$
$$a\beta(z\alpha - 1) = a\beta z\alpha - a\beta = \bar{a}(z\alpha - 1)\, a\alpha - \bar{a}(z\alpha - 1)$$
$$= \bar{a}(a\alpha - 1)\,(z\alpha - 1).$$

Therefore $a\beta = \bar{a}\,(a\alpha - 1)$.

Let $\partial = (\alpha, \beta, \gamma)$ and $\partial' = (\alpha', \beta', \gamma')$ be elements in $\mathscr{A}$. For $a \in P_D$ and $\bar{a} \in A$, define $a(\alpha + \alpha') = a\alpha\, a\alpha'$, $a(\beta + \beta') = a\beta + a\beta'$ and $\bar{a}(\gamma + \gamma') = \bar{a}\gamma + \bar{a}\gamma'$. Then $\partial\partial' = (\alpha\alpha'. \alpha\beta' + \beta\gamma', \gamma\gamma')$, where $\alpha\beta' + \beta\gamma'$ is a crossed homomorphism of P_D into A with respect to $\alpha\alpha'$. Let X be the function that maps each element $\bar{a} \in A$ upon $1 \in D$. Then ∂ can be represented by the matrix

$$\begin{bmatrix} \alpha & \beta \\ X & \gamma \end{bmatrix}$$

and the product $\partial\partial'$ is equivalent to the product of the corresponding matrices. The mapping of ∂ onto the unique semi-o-automorphism (α, γ) of A that is determined by α and γ is a homomorphism of $\mathscr{A}$ onto S with kernel,

$$\mathscr{K} = [\partial = (\alpha, \beta, \gamma) \in \mathscr{A} : \alpha \text{ and } \beta \text{ are the identity automorphisms}].$$

From VII it is easy to show that $\mathscr{K}$ is isomorphic to $(A, +)$.

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ON ORDERED VECTOR SPACES

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In this note all possible methods of ordering a (left) vector space A of dimension 2 over a division ring D are classified. The results and notation developed in [1] will be used throughout. If A is ordered so that it has exactly one proper convex subspace, then the ordering is a T-ordering, and any two such orderings with respect to the same ordering of D are automorphic ([1] Corollary 3.2). Suppose that A is ordered and has no proper convex subspaces. If the associated order of D is Archimedean, then without loss of generality D is a *proper* subfield of the real field R ([1], Theorem 3). There exists an element $q \in R \setminus D$ such that $(a, b) \in A$ is positive if and only if $a + bq$ is positive in R ([1], Theorem 10). Thus to complete this classification, it suffices to determine the methods of ordering A so that it has no proper convex subspaces and the associated order of D is non-Archimedean. There are two ways of doing this: Method I (see [1] or the proof of the following theorem) and a generalization of Method II which we now state. As in [1], let v be the natural order valuation of D, and let Π be the associated value group with identity θ.

METHOD III. Suppose that D has a non-Archimedean order, and let $\mathscr{L}$, $\mathscr{U}$ be a cut in D. That is, $\mathscr{L}$ and $\mathscr{U}$ are disjoint subsets of D such that $\mathscr{L} \cup \mathscr{U} = D$, and $a \in \mathscr{L}$, $b \in \mathscr{U}$ imply $a < b$. Define $(a, b) \in A$ positive if

(1) $v(a) > v(b)$ and $a > 0$, or

(2) $v(a) < v(b)$ and $b > 0$, or

(3) $v(a) = v(b)$, $a > 0$ and $a^{-1} b \in \mathscr{U}$, or

(4) $v(a) = v(b)$, $b > 0$ and $a^{-1} b \in \mathscr{L}$.

The proof that this definition does order A is long and more or less straightforward. The most tedious part is showing that the set

P of all positive elements in A is closed with respect to addition. We illustrate by proving one of the more intricate of the ten possible cases. Suppose that $v(a) = v(b)$, $a > 0$ and $a^{-1} b \in \mathcal{U}$, and that $v(c) = v(d)$, $d > 0$ and $c^{-1} d \in \mathcal{L}$. Then

$$(*)\ c^{-1} d < a^{-1} b,\ ac^{-1} < bd^{-1}$$

and

$$(a + c)c^{-1} = ac^{-1} + 1 < bd^{-1} + 1 = (b + d)d^{-1}.$$

We must show that $X \equiv (a+c, b + d) \in P$.

First assume that $v(a) = v(b) < v(c) = v(d)$. Then $v(b + d) = v(a + c)$ and $b + d > 0$. Thus $X \in P$ if $a + c \geqslant 0$. If $a + c < 0$ then by (*), $(a + c)^{-1} (b + d) < c^{-1} d \in \mathcal{L}$. Hence $X \in P$. Next assume that $v(a) = v(b) > v(c) = v(d)$. Then $v(a + c) = v(b + d)$ and $a + c > 0$. Thus $X \in P$ if $b + d \geqslant 0$. If $b + d < 0$, then $b < 0$. Moreover, $db^{-1} < ca^{-1}$. For if $c < 0$, this follows from (*), and if $c > 0$, then $db^{-1} < 0 < ca^{-1}$. Thus $(b + d)b^{-1} < (a + c)a^{-1}$. Hence $(a + c)^{-1} (b + d) > a^{-1}b \in \mathcal{U}$.

Finally assume that $v(a) = v(b) = v(c) = v(d)$.

Case I. $v(a + c) = v(b + d)$. If $a + c < 0$, then $c < 0$ and from (*), $(a + c)^{-1} (b + d) < c^{-1} d \in \mathcal{L}$. Moreover, since $(a + c)^{-1}$ and $c^{-1} d$ are negative, $b + d > 0$. Suppose that $a + c > 0$. Then $X \in P$ if $b + d > 0$. If $b + d < 0$, then $b < 0$ and since $c^{-1} d < a^{-1} b$, $c < 0$. Thus as above $(a + c)^{-1} (b + d) > a^{-1} b \in \mathcal{U}$.

Case II. $v(a + c) > v(b + d)$. Suppose that $a + c < 0$. Then $c < -a < 0$. Hence by (*), $0 < (a + c)c^{-1} < (b + d) d^{-1}$. But $v[(a + c) c^{-1}] = v(a + c) - v(c) > v(b + d) - v(d) = v[(b + d) d^{-1}]$, a contradiction.

Case III. $v(a + c) < v(b + d)$. Suppose that $b + d < 0$. Then by (*), $(a + c) c^{-1} < (b + d) d^{-1} < 0$. But $v[(a + c) c^{-1}] = v(a + c) - v(c) < (v\, b + d) - v(d) = v[(b + d) d^{-1}]$, a contradiction.

Lemma. *A has a proper convex subspace if and only if there exists a negative element $q \in D$ such that $v(q) = \theta$ and q is the least element in $\mathcal{U}$ or q is the greatest element in $\mathcal{L}$.*

PROOF. Suppose that $\mathscr{U}$ has no least element and that $\mathscr{L}$ has no greatest element. Then A has no proper convex subspace. For let S be a non-zero convex subspace of A, and consider $(a, b) \in S \cap P$ and $(c, d) \in A \cap P$. If $v(a) > v(b)$ and $a > 0$, then for large positive $f \in D$, $v(fa) = v(f) + v(a) > v(f) + v(b) = v(fb) > \max [v(c), v(d)]$, and $fa - c > 0$. Thus $v(fa - c) > v(fb - d)$, and $(fa - c, fb - d)$ is positive. Hence $(c, d) < (fa, fb) \in S$. Therefore $(c, d) \in S$. If (a, b) satisfies (3), then there exists an element $u \in \mathscr{U}$ such that $a^{-1} b > u$, because $\mathscr{U}$ has no least element. For large positive $f \in D$, $v(fa-c) = v(fb - d)$, $fa - c > 0$ and since $b > au, f(b-au) > d - cu$. Thus $fb - d > (fa - c) u$. Hence $(fa - c)^{-1} (fb-d) > u$. Similarly if (a, b) satisfies (2) or (4), then $(c, d) \in S$. Therefore $S = A$.

Next suppose that q is the least element in $\mathscr{U}$. If $q \geqslant 0$, then conditions (1) to (4) are equivalent to (1) and (5): $v(a) \leqslant v(b)$ and $b > 0$. For if (a, b) satisfies (1), (2) or (4), then clearly it satisfies (1) or (5). If (a, b) satisfies (3), then $a^{-1} b \geqslant q \geqslant 0$. Thus $b > 0$. Hence (a, b) satisfies (5). Conversely, suppose that $v(a) = v(b)$ and $b > 0$. If $a > 0$, then (a, b) satisfies (3) or (4). If $a < 0$, then $a^{-1} b < 0 \leqslant q$. Hence (a, b) satisfies (4). Thus it is clear that A has no proper convex subspaces.

If $q < 0$ and $v(q) < \theta$, then, once again, conditions (1) to (4) are equivalent to (1) and (5). If $q < 0$ and $v(q) > \theta$, then conditions (1) to (4) are equivalent to (2) and (6): $v(a) \geqslant v(b)$ and $a > 0$. Thus in either case A has no proper convex subspaces.

If $q < 0$ and $v(q) = \theta$, then $D(1, q)$ is the proper convex subspace of A. It suffices to show that $(0, 0) < (a, b) < (1, q)$ implies $(a, b) \in D(1, q)$. Thus suppose that (a, b) and $(1 - a, q - b)$ are positive.

First assume that $v(a) > v(b)$ and $a > 0$. If $v(a) > \theta$, then $v(1 - a) = v(a) > v(b - q)$ and $1 - a < 0$, a contradiction. If $v(a) \leqslant \theta$, then $v(1 - a) \leqslant v(q - b) = \theta$ and $q - b < 0$. Thus $v(1 - a) = v(q - b)$, $q - b < 0 < 1 - a$. If $b \geqslant 0$, then $qa < 0 \leqslant b$. If $b < 0$, then $qa < b$, since $v(qa) = v(a) > v(b)$. Thus we have $q - b < q - qa$

$= q(1 - a)$. Hence $(1 - a)^{-1}(q - b) < q$ and $q - b < 0$, a contradiction.

Next assume that $v(a) = v(b)$, $a > 0$ and $a^{-1}b \geqslant q$. Then $b \geqslant aq$. If $b = aq$, then $(a, b) = a\,(1, q) \in D\,(1, q)$. Suppose that $b > aq$. Then $q - b < (1 - a)\,q$. If $v(1 - a) = v[\,(1 - a)\,q\,] < v(q - b)$, then $q - b > 0$. Hence $(1 - a)\,q < q - b$, a contradition. If $v(1 - a) = v[(1 - a)\,q] > v(q - b)$, then $1 - a > 0$. Hence $(1 - a)\,q < q - b$, a contradiction. Finally suppose that $v(1 - a) = v(q - b)$. If $1 - a < 0$, then $q - b > 0$, and $(1 - a)^{-1}(q - b) > q$, a contradiction. If $q - b < 0$, then $1 - a > 0$ and $(1 - a)^{-1}(q - b) < q$, a contradiction. Similarly if (a, b) satisfies (2) or (4), we get a contradiction. Therefore $(a, b) \in D\,(1, q)$.

By an almost identical proof it follows that if q is the greatest element in $\mathscr{L}$, then A has a proper convex subspace if and only if *$q < 0$ and $v(q) = \theta$, and this subspace is $D(1, q)$.* This completes the proof of the lemma.

If $\mathscr{U}$ is the null set, then conditions (1) to (4) are equivalent to (1) and (5), and if $\mathscr{L}$ is the null set, then conditions (1) to (4) are equivalent to (2) and (6). In either case A^{θ}/A_{θ} has an order induced by the ordering of A, where

$$A^{\theta} = \{(a, b) \in A : \max\,[v(a), v(b) \leqslant \theta\}$$

and

$$A_{\theta} = \{(a, b) \in A : \max\,[v(a), v(b)] < \theta\}.$$

Thus the ordering of A is an N-ordering ([1], Corollary 9.2) and we have

COROLLARY. *If q is the least element of $\mathscr{U}$ or the greatest element of $\mathscr{L}$ and $q \geqslant 0$ or $v(q) \neq \theta$, then the ordering of A is an N-ordering. Also if $\mathscr{L}$ or $\mathscr{U}$ is the null set, then the ordering of A is an N-ordering.*

In the proof of the following theorem $(a, b) \ll (c, d)$ means that (a, b) is positive and $n(a, b) < (c, d)$ for the integers n, and $(a, b) \sim (c, d)$ means that (a, b) and (c, d) are positive and there exist

positive integers m, n such that $m(a, b) > (c, d)$ and $n(c, d) > (a, b)$. Levi [2] shows that $\sim$ is an equivalence relation and that $\ll$ induces an ordering of the equivalence classes of positive elements. Note that $a \ll b$ in D if and only if $v(a) < v(b)$. We shall also use the fact that if P and P' are the sets of positive elements for two orderings of A, and $P \subseteq P'$, then the two orderings are the same.

THEOREM. *Suppose that A is ordered so that it has no proper convex subspaces and the associated order of D is non-Archimedean. Then the ordering of A is the result of applying Methods I or III.*

PROOF. Choose a positive basis (1, 0) and (0, 1) for A. Then $(a, b) \in A$ is positive (notation $(a, b) > \bar{0}$) if $a > 0$ and $b > 0$ or $ab = 0$ and $a + b > 0$. For each element γ in Π, pick a positive element $d \in D$ for which $v(d) = \gamma$. Let S be the set of all these representatives d of Π, and define

$$L = \{v(d) : d \in S \text{ and } (0, d) \ll (1, 0)\},$$

and

$$U = \{v(d) : d \in S \text{ and } (1, 0) \ll (0, d\}.$$

By the properties of $\ll$ and $\sim$, it follows that L and U are independent of the choice of the representatives d of Π and they are non-void. Moreover, either L, U is a cut in Π or there exists an element $\alpha \in \Pi$ such that L, U is a cut in $\Pi \setminus \{\alpha\}$. If L, U is a cut in Π, then $(a, b) \in A$ is positive if

(a) $ab = 0$ and $a + b > 0$, or

(b) $a > 0 \neq ab$ and $v(a^{-1}b) \in L$, or

(c) $b > 0 \neq ab$ and $v(a^{-1}b) \in U$.

Thus A is ordered by Method I. For assume that (a, b) satisfies (b). If $b > 0$, then $(a, b) > \bar{0}$. If $b < 0$, then $-a^{-1}b > 0$ and $(0, -a^{-1}b) \ll (1, 0)$. Hence $\bar{0} < (1, a^{-1}b)$ and $\bar{0} < (a, b)$. Similarly $(a, b) > \bar{0}$ if (a, b) satisfies (c).

Finally suppose that L, U is a cut in $\Pi \setminus \{\alpha\}$. Then without loss of generality $\alpha = \theta$. For we can use the basis (1, 0) and $(0, a)$, where $a > 0$ and $v(a) = \alpha$. Thus $L = \{\gamma \in \Pi : \gamma < \theta\}$ and $U = \{\gamma \in \Pi : \gamma > \theta\}$. That is, $(1, 0) \ll (0, d)$ if and only if $0 < d$ and $v(d) > \theta$.

Let $\mathscr{L} = \{d \in D : (1, d) < \bar{0}\}$ and $\mathscr{U} = \{d \in D : (1, d) > \bar{0}\}$. Then it is easy to show that $\mathscr{L}, \mathscr{U}$ is a proper cut in D. To complete the proof it suffices to show that if $(a, b) \in A$ satisfies one of the conditions (1) to (4) of Method III, then $(a, b) > \bar{0}$. If $v(a) > v(b)$ and $a > 0$, then $\theta > -v(a) + v(b) = v(a^{-1}b) \in L$. If $b \geqslant 0$, then clearly $(a, b) > \bar{0}$. If $b < 0$, then $-a^{-1}b > 0$ and $(0, a^{-1}b) \leqslant (1, 0)$. Thus $\bar{0} < a(a, a^{-1}b) = (a,b)$. Next assume that (a, b) satisfies (3). Then $(1, a^{-1}b) > \bar{0}$, hence $(a, b) > \bar{0}$. Similarly if (a, b) satisfies (2) or (4), then $(a, b) > \bar{0}$.

COROLLARY. *If L, U is a cut in Π such that L has a greatest element or U has a least element, then the ordering of A is an N-ordering.*

PROOF. We prove for the case where α is the least element of U. Without loss of generality $\alpha = \theta$. For we can use the basis $(1, 0)$ and $(0, d)$, where $0 < d$ and $v(d) = \alpha$. Thus $L = \{\gamma \in \Pi : \gamma < \theta\}$ and $U = \{\gamma \in \Pi : \gamma \geqslant \theta]$. It follows that $(a, b) > \bar{0}$ if and only if (a ,b) satisfies (1) or (5). Therefore the ordering is an N-ordering ([1], Corollary 9.2).

If B is an ordered vector space over D with no proper convex subspaces, then the subspaces of B have no proper convex subspaces. Thus the subspaces of B of dimension 2 are ordered by Methods I or III if the associated order of D is non-Archimedean, and are isomorphic to a subspace of $\bar{R}$ if D is Archimedean, where $\bar{R}$ is the naturally ordered vector space of the reals over a subfield that is order isomorphic to D. There are various ways of generalizing these methods so that they apply to finite dimensional vector spaces, but the author has not been able to give complete classification of all possible orderings of a vector space of dimension 3.

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RIEMANN SUMMABILITY OF CAUCHY PRODUCTS AND TRANSLATES OF SERIES†

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1. Introduction. It is the purpose of this note to resolve a contradiction in published accounts of properties of Riemann methods for evaluation of series. Let p be a positive integer. A given series $u_0 + u_1 + u_2 + \ldots$ is said to be evaluable to s by the Riemann method $R(p)$ of order p if the series in

$$\sigma(p, t) = \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt}\right)^p u_k \tag{1.1}$$

converges over some interval $0 < t < t_0$ to a function $\sigma(p, t)$ such that $\sigma(p, t) \to s$ as $t \to 0$.

The following assertion was given without indication of a method of proof by Vignaux [2].

ASSERTION 1.2. *If $\Sigma\, u_n$ is evaluable $R(p)$ to u and if $\Sigma\, v_n$ converges to v and $\Sigma\, |v_n| < \infty$, then the Cauchy product series Σw_n with terms defined by*

$$w_n = u_0 v_n + u_1 v_{n-1} + \cdots + u_{n-1} v_1 + u_n v_0 \tag{1.21}$$

is evaluable $R(p)$ to uv.

If this assertion is valid and if the series $u_0 + u_1 + u_2 + \ldots$ is evaluable $R(p)$, then the fact that the series $0 + 1 + 0 + 0 + 0 + \ldots$ is absolutely convergent would imply that the Cauchy product series $0 + u_0 + u_1 + u_2 + \ldots$ is evaluable $R(p)$. But on the other hand Kuttner [1] has given, without details of supporting arguments, an example of a series $u_0 + u_1 + u_2 + \ldots$ which is said to be evaluable $R(2)$ while the two series $0 + u_0 + u_1 + \ldots$

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and $u_1 + u_2 + u_3 + \dots$ are said to be non-evaluable $R(2)$. Thus we have a contradiction which indicates that a mistake has been made, perhaps in a delicate step involving change of order of summation in multiple series. As we shall see, the example of Kuttner is correct and therefore Assertion 1.2 is false.

We shall prove that corresponding to each positive integer p, there is an extensive class of series Σu_k such that, for each series Σu_k in the class, the series $u_0 + u_1 + u_2 + \dots$ is evaluable $R(p)$ while both of the translates $0 + u_0 + u_1 + \dots$ and $u_1 + u_2 + u_3 + \dots$ are non-evaluable $R(p)$. For the case in which $p = 2, 4, 6, \dots$ the examples are given in § 2 and the proofs are given in §§ 3 and 4. In the case $p = 1, 3, 5, \dots$ the examples are given in § 5 and the proofs are given in §§ 6 and 7. In the case $p = 2$, the examples of §2 reduce to those given by Kuttner.

The proofs depend upon the introduction of Abel power series transforms of series and upon the use of associated Abelian and Tauberian theorems.

2. Examples when p is even. Let p be an even positive integer. Let a_n and b_n be sequences of non-zero numbers for which $\Sigma|a_n| < \infty$, $\pi/2 > b_1 > b_2 > \dots$, and $\lim b_n = 0$.

Let

$$u_k = k^{p-1} \sum_{j=1}^{\infty} a_j\, b_j^p \sin kb_j, \qquad k = 0, 1, 2, \dots \tag{2.1}$$

and observe that $u_0 = 0$. In § 3 it is proved that the series $u_0 + u_1 + u_2 + \dots$ is evaluable $R(p)$ to 0.

In § 4 we impose on the sequence b_n an additional condition by requiring that if j and n are positive integers, then

$$b_j - (r/p)\, b_n \neq 0, \qquad r = 2, 4, 6, \dots, p-2. \tag{2.2}$$

In the case $p = 2$, this additional condition is vacuous and may be eliminated. In the case $p > 2$ and b_n decreases slowly, (2.2) imposes a mild restriction on the ratios b_{n+1}/b_n, $b_{n+2}/b_n, \dots$. The condition (2.2) is certainly satisfied whenever b_n decreases so

rapidly that $b_{n+1} < (2/p)\, b_n$. The condition (2.2), as well as the condition (5.2) appearing later, can be put in the form

$$b_j + (p - 2m)^{p-1}\, b_n \neq 0, \quad 0 \leqslant m < p. \tag{2.3}$$

It is proved that neither $0 + u_0 + u_1 + \dots$ nor $u_1 + u_2 + \dots$ is evaluable $R(p)$ because there is no interval $0 < t < t_0$ over which $R(p)$ transforms are convergent. Actually a more general result is proved which implies that $0 + 0 + u_1 + \dots$; $u_2 + u_3 + \dots$ and translates of higher orders are non-evaluable $R(p)$.

3. Proof that $u_0 + u_1 + \dots$ is evaluable $R(p)$ to 0 ; p even. We shall show that, when p is even, the series $u_0 + u_1 + \dots$ with terms defined by (2.1) is evaluable $R(p)$ to 0 by proving that the series in

$$\sigma(t) = \sum_{k=1}^{\infty} \frac{\sin^p kt}{kt^p} \sum_{j=1}^{\infty} a_j\, b_j^p \sin kb_j \tag{3.1}$$

converges when $t > 0$ to a function $\sigma(t)$ for which $\sigma(t) \to 0$ as $t \to 0$. When $0 < x < 1$ the series in

$$\sigma(t, x) = \frac{1}{t^p} \sum_{k=1}^{\infty} \frac{\sin^p kt}{k}\, x^k \sum_{j=1}^{\infty} a_j\, b_j^p \sin kb_j \tag{3.2}$$

converges absolutely and we can invert the order of summation to obtain

$$\sigma(t, x) = \frac{1}{t^p} \sum_{j=1}^{\infty} a_j\, b_j^p \sum_{k=1}^{\infty} \frac{\sin^p kt \sin kb_j}{k}\, x^k. \tag{3.3}$$

Expanding the last factor of the formula $\sin^p \theta = (2i)^{-p}(e^{i\theta} - e^{-i\theta})^p$ by the binomial formula leads to the formula

$$\sin^p \theta = \frac{1}{(2i)^p} \sum_{m=0}^{p} (-1)^m \binom{p}{m} e^{i(p-2m)\theta}. \tag{3.31}$$

Since p is even, we can multiply (3.31) by $e^{i\phi}$ and take imaginary parts to obtain

$$\sin^p \theta \sin \phi = \frac{1}{(2i)^p} \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sin\, [(p - 2m)\, \theta + \phi]. \tag{3.32}$$

Using (3.32) in (3.3) gives

$$\sigma(t, x) = \frac{1}{(2i)^p\, t^p} \sum_{j=1}^{\infty} a_j b_j^p f_j\,(t, x), \tag{3.4}$$

where

$$f_j\,(t,x) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sum_{k=1}^{\infty} \frac{\sin k[(p-2m)t + b_j]}{k}\, x^k. \tag{3.41}$$

Using the fact that $\sin k\theta$ is the imaginary part of $e^{ik\theta}$, it is easy to show that the formula

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} x^k = \int_0^x \frac{\sin\theta}{1 - y\cos\theta + y^2}\, dy = \int_0^x \frac{\sin\theta}{\sin^2\theta + (y-\cos\theta)^2}\, dy \tag{3.5}$$

is valid when θ is real and $0 \leqslant x \leqslant 1$. It is easy to evaluate the last integral in (3.4) in terms of inverse tangents and to show that the members of (3.5) are uniformly bounded over $-\infty < \theta < \infty$ and $0 \leqslant x \leqslant 1$. Moreover,

$$\lim_{x\to 1} \sum_{k=1}^{\infty} \frac{\sin k\theta}{k} x^k = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k\theta} \tag{3.51}$$

when $-\infty < \theta < \infty$ and

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \frac{\pi - \theta}{2}, \quad 0 < \theta < 2\pi, \tag{3.52}$$

these facts being very well known. It follows that the functions $f_j\,(x, t)$ in (3.41) are uniformly bounded and that

$$\lim_{x\to 1} f_j\,(t, x) = f_j\,(t, 1). \tag{3.53}$$

For each fixed $t > 0$ the series in (3.4) is therefore uniformly convergent over $0 \leqslant x \leqslant 1$ and we can let $x \to 1$ termwise in (3.4) to obtain

$$\lim_{x\to 1} \sigma(t, x) = \frac{1}{(2i)^p t^p} \sum_{j=1}^{\infty} a_j b_j^p f_j\,(t, 1). \tag{3.6}$$

Representing the right member of (3.1) *momentarily in the form* Σv_k we see that, for each fixed $t > 0$, $kv_k = O(1)$ and we see from (3.2) *that* $\sigma(t, x)$ is the Abel power series transform of Σv_k. Hence (3.6) implies that Σv_k is evaluable A to the right member of (3.6),

and a fundamental Tauberian theorem implies that Σv_k converges to the right member of (3.6). Therefore the series in (3.1) does in fact converge when $t > 0$ to a function $\sigma(t)$ and

$$\sigma(t) = \frac{1}{(2i)^p t^p} \sum_{j=1}^{\infty} a_j b_j^p f_j(t), \tag{3.7}$$

where

$$f_j(t) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sum_{k=1}^{\infty} \frac{\sin k\,[(p-2m)\,t + b_j]}{k}. \tag{3.71}$$

Suppose now that t is a fixed number for which $0 < pt < \pi/2$. For those values of j for which $b_j > pt$, we have

$$0 < (p - 2m)\,t + b_j < 2\pi, \quad 0 \leqslant m \leqslant p, \tag{3.72}$$

and it follows from (3.71) and (3.52) that

$$f_j(t) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \frac{\pi - [(p-2m)t + b_j]}{2} = 0. \tag{3.73}$$

This and (3.7) and the fact that there is a constant M for which $|f_j(t)| \leqslant M$ for each j and t imply that if $0 < t < \pi/2p$ then

$$\begin{aligned} |\sigma(t)| &\leqslant \frac{1}{2^p t^p} \sum_{b_j \leqslant pt} |a_j|\, b_j^p M \\ &\leqslant \frac{1}{2^p t^p} \sum_{b_j \leqslant pt} |a_j|\, p^p t^p M = \frac{Mp^p}{2^p} \sum_{b_j \leqslant pt} |a_j|. \end{aligned} \tag{3.8}$$

This implies that $\sigma(t) \to 0$ as $t \to 0$ and completes the proof that the series $u_0 + u_1 + \ldots$ is evaluable $R(p)$ to 0.

That the series Σu_k with terms defined by (2.1) should, when p is an even positive integer, be evaluable to 0 by reasonable methods by which it is evaluable is not surprising because of the completely formal calculations

$$\begin{aligned} \sum_{k=1}^{\infty} u_k &= \sum_{k=1}^{\infty} k^{p-1} \sum_{j=1}^{\infty} a_j b_j^p \sin kb_j \\ &= \sum_{j=1}^{\infty} a_j b_j^p \sum_{k=1}^{\infty} k^{p-1} \sin kb_j \end{aligned} \tag{3.9}$$

and

$$\sum_{k=1}^{\infty} k^{p-1} \sin kx = \pm \frac{d^p}{dx^p} \sum_{k=1}^{\infty} \frac{\sin kx}{k} \tag{3.91}$$

$$= \pm \frac{d^p}{dx^p} \frac{\pi - x}{2} = 0.$$

4. Proof that $0 + u_0 + u_1 + \ldots$ **and** $u_1 + u_2 + \ldots$ **are non-evaluable** $R(p)$, p **even.** Supposing as in § 3 that p is an even positive integer, that $u_0 + u_1 + \ldots$ is the series with terms defined by (2.1), and that (2.2) holds, we prove that neither $0 + u_0 + u_1 + \ldots$ nor $u_1 + u_2 + \ldots$ is evaluable $R(p)$ by showing that there is no interval $0 < t < t_0$ over which the $R(p)$ transforms

$$\sum_{k=2}^{\infty} \left(\frac{\sin kt}{kt}\right)^p u_{k-1}, \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt}\right)^p u_{k+1} \tag{4.1}$$

of the latter series are convergent. We shall prove more than is required by proving that if h is a real number for which $h \neq 0$, then there is no interval $0 < t < t_0$ over which

$$\sum_{k=1}^{\infty} \frac{\sin^p kt}{k^p} (k+h)^{p-1} \sum_{j=1}^{\infty} a_j b_j^p \sin (k+h) b \tag{4.2}$$

is convergent. Since the series

$$\sum_{k=1}^{\infty} \frac{\sin^p kt}{k^p} [(k+h)^{p-1} - k^{p-1}] \sum_{j=1}^{\infty} a_j b_j^p \sin (k+h) b_j \tag{4.21}$$

converges when $t > 0$, it is sufficient to prove that if $h \neq 0$, then there is no interval $0 < t < t_0$ over which the series

$$\sum_{k=1}^{\infty} \frac{\sin^p kt}{k} \sum_{j=1}^{\infty} a_j b_j^p \sin (k+h) b_j \tag{4.3}$$

is convergent. We shall establish the result by proving that (4.3) is divergent whenever $h \neq 0$ and $t = p^{-1} b_n$, where n is an integer so great that $0 < p^{-1} b_n < \pi$.

Since (4.3) is a series of the form Σv_k where $kv_k = O(1)$, it follows that (4.3) converges to $f(t)$ if and only if

$$\lim_{x \to 1} f(t, x) = f(t), \tag{4.4}$$

where

$$f(t, x) = \sum_{k=1}^{\infty} \frac{\sin^p kt}{k} x^k \sum_{j=1}^{\infty} a_j b_j^p \sin (k + h) b_j. \tag{4.41}$$

When $0 < x < 1$ the series in (4.41) converges absolutely and we can invert the order of summation and use (3.32) to obtain a formula for $f(t, x)$. Putting $t = p^{-1} b_n$ in the resulting formula we get

$$f(p^{-1} b_n, x) = \frac{1}{(2i)^p} \sum_{m=0}^{\infty} (-1)^m \binom{p}{m} \sum_{j=1}^{\infty} a_j f_j (m, x), \tag{4.5}$$

where

$$f_j (m, x) = b_j^p \sum_{k=1}^{p} \frac{\sin\{k[b_j + (p - 2m) p^{-1} b_n] + hb_j\}}{k} x^k. \tag{4.51}$$

The method by which (3.5) was established shows that the formula

$$\sum_{k=1}^{\infty} \frac{\sin (k\theta + \lambda)}{k} x^k = \int_0^x \frac{\sin (\theta + \lambda) - y \sin \lambda}{1 - 2y \cos \theta + y^2} dy \tag{4.6}$$

is valid when θ and λ are real and $0 < x < 1$. Since

$$|\sin (\theta + \lambda) - y \sin \lambda| \leqslant 2 \tag{4.61}$$

and

$$1 - 2y \cos \theta + y^2 = \sin^2 \theta + (y - \cos \theta)^2 \geqslant \sin^2 \theta \tag{4.62}$$

when θ and λ are real and $0 \leqslant y < 1$, it follows from (4.6) that

$$\left| \sum_{k=1}^{\infty} \frac{\sin (k\theta + \lambda)}{k} x^k \right| \leqslant \frac{2}{\sin^2 \theta}. \tag{4.63}$$

Taking the special case in which $m = p/2$, we find from (4.51) and (4.6) that

$$|f_j (p/2, x)| = b_j^p \left| \sum_{k=1}^{\infty} \frac{\sin (kb_j + hb_j)}{k} x^k \right|$$
$$\leqslant b_j^{p-2} \left(\frac{b_j}{\sin b_j}\right)^2 \leqslant \left(\frac{\pi}{2}\right)^{p-2} \left(\frac{\pi}{2}\right)^2 = \left(\frac{\pi}{2}\right)^p. \tag{4.7}$$

When $0 \leqslant m < p$ and $m \neq p/2$ the hypothesis (2.2) implies that

$$b_j + (p - 2m)p^{-1} b_n \neq 0, j = 1, 2, 3, \ldots \tag{4.71}$$

and since

$$\lim_{j\to\infty} [b_j + (p - 2m)p^{-1} b_n] = (p - 2m) p^{-1} b_n \neq 0, \tag{4.72}$$

there must be a positive constant δ_m such that

$$\delta_m < |b_j + (p - 2m)p^{-1}b_n| < \pi/2, \quad j = 1, 2, 3, \ldots \tag{4.73}$$

and hence

$$|f_j(m, x)| \leqslant b_j^p (\sin \delta_m)^{-2} \leqslant M_m \tag{4.74}$$

for some constant M_m. It follows from (4.5), (4.7) and (4.74) that there is a function $F_n(x)$ uniformly bounded over $0 \leqslant x < 1$ for which

$$f(p^{-1} b_n, x) = F_n(x) + \frac{1}{(2i)^p} \sum_{j=1}^{\infty} a_j f_j(p, x), \tag{4.8}$$

where, as (4.51) shows,

$$f_j(p, x) = b_j^p \sum_{k=1}^{\infty} \frac{\sin [k(b_j - b_n) + hb_j]}{k} x^k. \tag{4.81}$$

Since there is a positive constant δ for which $|b_j - b_n| > \delta$ when $j \neq n$ it follows from (4.63) that, when $j \neq n$,

$$|f_j(p, x)| \leqslant b_j^p (\sin \delta)^{-2} \leqslant M \tag{4.82}$$

for some constant M. It follows from this and (4.8) that there is a function $G_n(x)$ bounded over $0 \leqslant x < 1$ for which

$$f(p^{-1} b_n, x) = G_n(x) + \frac{a_n b_n^p}{(2i)^p} \sum_{k=1}^{\infty} \frac{\sin hb_n}{k} x^k, \tag{4.83}$$

the last term in (4.83) being the term in the sum in (4.8) for which $j = n$. Thus

$$f(p^{-1} b_n, x) = G_n(x) + \frac{a_n b_n^p \sin hb_n}{(2i)^p} \log \frac{1}{1-x} \tag{4.9}$$

and it follows that if n is so great that $0 < |hb_n| < \pi$ then

$$\lim_{x\to 1} |f(t, x)| = \infty \tag{4.91}$$

when $t = p^{-1} b_n$. It follows that when $t = p^{-1} b_n$ and n is sufficiently great, the limit in (4.4) does not exist and the series (4.3) is divergent. This completes the proofs of the results of § 2 concerning $R(p)$ when p is even.

5. Examples when p is odd. Let p be an odd positive integer. Let a_n and b_n be sequences of non-zero numbers for which $\Sigma\,|a_n|<\infty$, $\pi/2 > b_1 > b_2 > \ldots$, and $\lim b_n = 0$. Let $u_0 = 0$ and let

$$u_k = k^{p-1} \sum_{j=1}^{\infty} a_j b_j^p \cos kb_j, \qquad j = 1, 2, 3, \ldots. \tag{5.1}$$

In § 6 it is proved that the series $u_0 + u_1 + \ldots$ is evaluable $R(p)$ to the value V_1 in (6.1) when $p = 1$ and to the value $V_p = 0$ when $p = 3, 5, 7, \ldots$.

In § 7 we impose on the sequence b_n an additional condition by requiring that if j and n are positive integers then

$$b_j - \frac{r}{p} b_n \neq 0, \qquad r = 1, 3, 5, \ldots, p-1. \tag{5.2}$$

In the case $p = 1$, this additional condition is vacuous. It is proved that neither $0 + u_0 + u_1 + \ldots$ nor $u_1 + u_2 + \ldots$ is evaluable $R(p)$ because there is no interval $0 < t < t_0$ over which their $R(p)$ transforms are convergent. Remarks similar to those of § 2 are relevent.

6. Proof that $u_0 + u_1 + \ldots$ is evaluable $R(p)$, p odd. We shall show that when p is odd the series $u_0 + u_1 + \ldots$ with terms defined by (5.1) is evaluable $R\,(p)$ to the value V_p, where

$$V_p = \frac{-1}{2} \sum_{k=1}^{\infty} a_j\, b_j^p \tag{6.1}$$

when $p = 1$, and $V_p = 0$ when $p = 3, 5, 7, \ldots$, by showing that $\sigma(t, 1) \to V_p$ as $t \to 0$, where

$$\sigma(t, x) = \sum_{k=1}^{\infty} \frac{\sin^p kt}{kt^p} x^k \sum_{j=1}^{\infty} a_j b_j^p \cos kb_j. \tag{6.2}$$

When $0 < x < 1$ we can invert the order of summation in (6.2) to obtain

$$\sigma(t,x) = \frac{1}{t^p} \sum_{j=1}^{\infty} a_j b_j^p \sum_{k=1}^{\infty} \frac{\sin^p kt \cos kb_j}{k} x^k. \tag{6.3}$$

Since p is now odd, we can multiply (3.31) by $e^{i\phi}$ and take real parts to obtain

$$\sin^p \theta \cos \phi = \frac{-2}{(2i)^{p+1}} \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sin [(p-2m)\theta + \phi]. \tag{6.31}$$

Using (6.31) in (6.3) we get

$$\sigma(t,x) = \frac{2}{(2i)^{p+1} t^p} \sum_{j=1}^{\infty} a_j b_j^p f_j(t,x), \tag{6.4}$$

where

$$f_j(t,x) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sum_{k=1}^{\infty} \frac{\sin k[(p-2m)t + b_j]}{k} x^k. \tag{6.41}$$

It follows by the argument included between (3.41) and (3.7) that

$$\sigma(t,1) = -\frac{-2}{(2i)^{p+1} t^p} \sum_{j=1}^{\infty} a_j b_j^p f_j(t), \tag{6.5}$$

where $f_j(t)$ is defined by (3.71). Suppose now that t is a fixed number for which $0 < pt < \pi/2$. For those values of j for which $b_j > pt$, (3.72) holds and consequently the first part of (3.73) holds so that

$$f_j(t) = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \frac{\pi - [(p-2m)t + b_j]}{2}. \tag{6.51}$$

In the case $p = 1$ we therefore have $f_j(t) = -t$ when $b_j > pt$, and in the case $p = 3, 5, 7, \dots$ we have $f_j(t) = 0$ when $b_j > pt$. Thus when $p = 1$ we find from (6.5) that

$$\sigma(t,1) = \frac{-1}{2} \sum_{b_j > t} a_j b_j + \frac{1}{2t} \sum_{b_j < t} a_j b_j f_j(t) \tag{6.6}$$

and in the case $p = 3, 5, 7, \dots$ we find from (6.5) that

$$\sigma(t,1) = \frac{-2}{(2i)^{p+1} t^p} \sum_{b_j < pt} a_j b_j^p f_j(t). \tag{6.61}$$

Considering separately the case where $p = 1$ and the case where $p = 3, 5, 7, \ldots$ we obtain the required conclusions by use of arguments similar to those in (3.8).

That the series Σu_k with terms defined by (5.1) should, when p is an odd positive integer, be evaluable to the stated value V_p by reasonable methods by which it is evaluable is not surprising because of the completely formal calculations

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} k^{p-1} \sum_{j=1}^{\infty} a_j b_j^p \cos kb_j \tag{67}$$

$$= \sum_{j=1}^{\infty} a_j b_j^p \sum_{k=1}^{\infty} k^{p-1} \cos kb_j$$

and, when $p = 1$,

$$\sum_{k=1}^{\infty} k^{p-1} \cos kx = \frac{d}{dx} \sum_{k=1}^{\infty} \frac{\sin kx}{k} \tag{6.71}$$

$$= \frac{d}{dx} \frac{\pi - x}{2} = -\tfrac{1}{2}$$

and, when $p = 3, 5, 7, \ldots$

$$\sum_{k=1}^{\infty} k^{p-1} \cos kx = \pm \frac{d^p}{dx^p} \sum_{k=1}^{\infty} \frac{\sin kx}{k} = 0. \tag{6.72}$$

7. **Proof that $0 + u_0 + u_1 + \ldots$ and $u_1 + u_2 + \ldots$ are non-evaluable $R(p)$, p odd.** Supposing as in § 6 that p is an odd positive integer, that $u_0 + u_1 + \ldots$ is the series with terms defined by (5.1), and that (5.2) holds, we prove that neither $0 + u_0 + u_1 \ldots$ nor $u_1 + u_2 + \ldots$ is evaluable $R(p)$ by showing that there is no interval $0 < t < t_0$ over which the $R(p)$ transforms (4.1) of the latter series are convergent. As in § 4, we prove more than is required by proving that if $h \neq 0$ then there is no interval $0 < t < t_0$ over which the series in

$$f(t, x) = \sum_{k=1}^{\infty} \frac{\sin^p kt}{k} x^k \sum_{j=1}^{\infty} a_j b_j^p \cos (k + h) b_j \tag{7.1}$$

converges when $x = 1$. We see as in § 4 that it is sufficient to prove that there is no interval $0 < t < t_0$ of values of t for which $\lim_{x \to 1} f(t, x)$ exists. When $0 < x < 1$ the series in (7.1) converges absolutely and we can invert the order of summation, use (6.31), and put $t = p^{-1} b_n$ in the result to obtain

$$f(p^{-1} b_n, x) = \frac{-2}{(2i)^{p+1}} \sum_{m=0}^{p} (-1)^m \binom{p}{m} \sum_{j=1}^{\infty} a_j f_j(m, x), \quad (7.2)$$

where

$$f_j(m, x) = b_j^p \sum_{k=1}^{\infty} \frac{\sin \{k[b_j + (p - 2m) p^{-1} b_n] + hb_j\}}{k} x^k. \quad (7.21)$$

The remainder of the proof is very much like the argument following (4.5) and (4 51), but is simplified by the fact that p is now odd and there is no integer m for which $p - 2m = 0$. If $0 \leqslant m < p$ and (5.2) and hence (4.71) holds, the argument preceding (4.74) shows that (4.74) holds and the proof is completed by the argument following (4.74).

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THE PROBLEM OF "TOTAL TRANSLATIVITY" FOR HAUSDORFF METHODS

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INTRODUCTION. The problem of the total translativity of Hölder and Cesàro methods has been considered earlier by Kuttner [3]. In the present note we are concerned with the more general problem of the total translativity of Hausdorff methods and we prove that a Hausdorff method, satisfying suitable restrictions, is totally translative if, and only if, it is the Cesàro method or a trivial modification of that method.

DEFINITIONS AND LEMMAS. A method of summability, say A, transforming $\{s_n\} = \{s_0, s_1, \ldots\}$ into $\{t_n\}$ is said to be *regular* when $s_n \to l$ implies $t_n \to l$, where l is a finite number. The method or the transformation is said to be *totally regular* if it has that the above property holds even when $l = \pm\infty$, $\{s_n\}$ being assumed to be real.

Consider now the sequence $\{\bar{s}_n\} = \{0, s_0, s_1, \ldots\}$. If the A-summability of $\{s_n\}$ to l (finite) implies the A-summability of $\{\bar{s}_n\}$ also to l, and conversely, then A is said to be *translative*. If the property still holds when $l = \pm\infty$, then A is defined to be *totally translative.*

It is well known (see, for example, Kuttner [3]), that the method (C, k) is totally translative, while the method (H, k) is not totally translative except for $k = 1$, when it is identical with the $(C, 1)$ method, and for $k = 0$, when its transformation reduces to the identity transformation. It may also be easily seen that the Euler method $E(p)$, defined as the Hausdorff method (H, μ_n) with $\mu_n = p^n$, is totally translative only for $p = 1$, when it reduces to the identity method.

We are now concerned with obtaining the most plausible form of the μ_n which will make the Hausdorff method (H, μ_n), subject to suitable restrictions, totally translative. These restrictions, to

quote Kuttner [3] who formulated them, *"while admittedly stringent, are satisfied by all the common methods"*.

We require the following definitions and lemmas.

LEMMA 1. *The sequence-to-sequence transformation defined by the triangular T-matrix A is totally regular if, and only if*

$$a_{nm} \geqslant 0, \ (m \geqslant m_0, \ \textit{all } n).$$

Lemma 1 is due to Hurwitz [2].

LEMMA 2. *Suppose that the sequence $\{t_n\}$ is given in terms of the sequence $\{s_n\}$ by a transformation of the form*

$$t_n = \sum_{m=0}^{n} a_{nm} s_m, \quad a_{nn} \neq 0.$$

Let the inverse transformation, which necessarily exists, be given by

$$s_n = \sum_{m=0}^{n} b_{nm} t_m.$$

Suppose also that both the transformations are regular. Then for both the transformations to be totally regular it is necessary that

$$a_{n,n-1} = 0, \textit{ for } n \geqslant n_0.$$

The proof of the above lemma is immediate from Lemma 1 and the relation

$$b_{n,n-1} = \frac{-a_{n,n-1}}{a_{nn} \cdot a_{n-1,n-1}}.$$

Let us start with a Hausdroff sequence-to-sequence method (H, μ_n) with $\mu_n \neq 0$ and defined by the matrix $(H, \mu_n) = H = (h_{nk})$,

$$h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k \ (n \geqslant k) \text{ and } h_{nk} = 0, \ (n < k).$$

For the method to be conservative it is necessary and sufficient that μ_n should be a *moment constant*†, generated by the function $\chi(t)$ say, of bounded variation in (0, 1), so that $\mu_n = \int_0^1 t^n \, d\chi(t)$. Then

†For definition and other properties, see Hardy [2, Chapter XI

there exists a *moment function* $T(z)$, regular for $\mathscr{R}z > 0$ and bounded and continuous for $\mathscr{R}\, z \geqslant 0$, defined by

$$T(z) = \int_0^1 t^z \, d\chi(t).$$

The transformation of a sequence $\{s_n\}$ into $\{t_n\}$, defined by the method A say, is said to be *multiplicative* ρ, if $\lim s_n = l$ implies $\lim t_n = \rho . l$, l assumed finite. Now, for a multiplicative Hausdorff transformation (H, μ_n), "*conditions A*" of Kuttner [4] are the following :

(i) *If* $K(z)$ *is defined as* $K(z) = \dfrac{T(z)}{T(z+1)}$, *where* $T(z)$ *is the moment function defined earlier, the function* $K(z)$ *is bounded for sufficiently large* $|z|$, *uniformly in* arg z, *for* $|\arg z| \leqslant \pi/2$;

(ii) *there are constants* a, A *and* $\lambda > 1$, *such that* $K(it) - a \in L_2$ *in* $(-\infty, -A)$ *and* (A, ∞) *and such that*

$$\left\{ \int_{-\infty}^{-A} + \int_{A}^{\infty} \right\} |K(it) - K(it+ih)|^2 \, dt \doteq o(h^\lambda)$$

for small h ;

(iii) *either* $T(z)$ *has no zeros on the line* $\mathscr{R}z = 1$ *or there is an integer* k *such that* $T(z)$ *has no zeros of order greater than* k *on this line such that*

$$\int_0^1 (1 + |\log t|^{k+1}) \, |d\chi(t)| < \infty.$$

We require also the following three more lemmas.

LEMMA 3. *Let* (H, μ_n) *be a multiplicative Hausdroff transformation and let* $K(z)$ *be defined as above. Suppose that* $\mu_n \neq 0$ *for all* n. *Suppose that* $K(z)$ *defined above is regular and bounded for* $\mathscr{R}z > 0$. *Suppose also that there is a constant* a *such that* $K(it) - a \in L^2(-\infty,\infty)$. *Suppose that either*

(i) $T(z)$ *has no zeros on the line* $\mathscr{R}z = 1$; *also for some* A *and some* $\lambda > 1$,

$$\left\{ \int_{-\infty}^{-A} + \int_{A}^{\infty} \right\} | k(it) - k(it + ih) |^2 \, dt = 0(h^\lambda)$$

for small h, *or*

(ii) $K(z)$ *is continuous for* $\mathscr{R}z \geqslant 0$ *and for some* $\lambda > 1$,

$$\int_{-\infty}^{\infty} | k(it) - k(it + ih) |^2 \, dt = O(h^\lambda).$$

Then $K(z)$ *is a Mellin transform and* (H, μ_n) *is translative.*

LEMMA 4. *Suppose that* (H, μ_n) *is a multiplicative Hausdorff transformation with* $\mu_n \neq 0$ *for all* n, *and that* (H, μ_n) *satisfies Conditions* A. *Then in order that* (H, μ_n) *should be translative it is necessary and sufficient that* $K(z)$ *should be regular for* $\mathscr{R}z > 0$ *and continuous for* $\mathscr{R}z \geqslant 0$.

Lemmas 3 and 4 are due to Kuttner ([4], Theorems 3, 4).

LEMMA 5. *Let* $\{t_n\}$ *and* $\{\bar{t}_n\}$ *denote the* (H, μ_n)*-transformations of* $\{s_n\} = \{s_0, s_1, \ldots\}$ *and* $\{\bar{s}_n\} = \{0, s_0, s_1, \ldots\}$ *respectively. Then* $\{t_n\}$ *can be defined as the transform of the sequence* $\{\bar{t}_{n+1}\}$ *by the matrix* $A \equiv (a_{nm})$, *where*

$$\left.\begin{aligned} a_{nm} &= \binom{n}{m} \Delta^{n-m} \left(\frac{\mu_m}{\mu_{m+1}} \right) - \binom{n}{m+1} \Delta^{n-m-1} \left(\frac{\mu_{m+1}}{\mu_{m+2}} \right) + \\ &\qquad + \binom{n}{m+1} \Delta^{n-m-1}(1), \ (n \geqslant m) \\ a_{nm} &= 0, \ (n < m). \end{aligned}\right\} \quad (1)$$

The proof of Lemma 5 follows from the proof of a known theorem ([4], Th. 2).

Now we are in a position to prove the following theorem.

THEOREM. *Let* $\lambda \equiv (H, \mu_n)$ $(\mu_n \neq 0$, *for all* $n)$ *define a multiplicative Hausdorff transformation, which satisfies the Conditions* A.

Then the method is totally translative if, and only if, it is the Cesaro's method or a multiple of it by a scalar.

PROOF. Let $\{t_n\}$ and $\{\bar{t}_n\}$ denote the transformations of $\{s_n\}$ and $\{\bar{s}_n\}$, defined as in Lemma 5. Then by Lemma 5, $\{t_n\}$ is the transform of $\{\bar{t}_{n+1}\}$ by the matrix A in (1), which is a normal triangular matrix.

Now in order that the method (H, μ_n) may be totally translative it is necessary that it must be translative. Since the method (H, μ_n) satisfies Conditions A, we have by Lemma 4 that (H, μ_n) is translative only if $K(z)$ is regular for $\mathscr{R}z > 0$ and continuous for $\mathscr{R}z \geqslant 0$, which, as in the proofs of Lemmas 4 and 3, implies that $K(z)$ is a Mellin transform, i.e. $\left(H, \dfrac{\mu_n}{\mu_{n+1}}\right)$ is a multiplicative method. This along with the fact that (H, μ_n) is, by hypothesis, a multiplicative method, implies as in the proof of a known theorem ([4], Th. 2) that the matrix A defined above by (1) and the matrix B, of the transformation inverse to the one defined by A are both T-matrices. Now, for the method (H, μ_n) to be totally translative it is clearly necessary and sufficient that both A and B, (which are regular), should both be totally regular. Consequently, by Lemma 2, it is necessary that

$$a_{n,n-1} = 0, \text{ for } n \geqslant n_0,$$

i.e.

$$n\,\Delta\left(\frac{\mu_{n-1}}{\mu_n}\right) - \frac{\mu_n}{\mu_{n+1}} + 1 = 0, \text{ for } n \geqslant n_0,$$

i.e.

$$n(\nu_{n-1} - \nu_n) - \nu_n + 1 = 0, \left(\text{where } \nu_n = \frac{\mu_n}{\mu_{n+1}}\right), \text{ for } n \geqslant n_0,$$

or

$$\begin{aligned}(n+1)\,\nu_n &= n\,\nu_{n-1} + 1 \\ &= (n-1)\,\nu_{n-2} + 2 \\ &= \dots \quad \dots \quad \dots \quad \dots \\ &= (n_0 + 1)\,\nu_{n_0} + (n - n_0), \ (n \geqslant n_0),\end{aligned}$$

i.e. for $n \geqslant n_0$,

$$\nu_n = [(n_0 + 1)\,\nu_{n_0} + (n - n_0)]/(n + 1).$$
$$= 1 + \frac{r}{n+1}, \quad r = (n_0 + 1)\,(\nu_{n_0} - 1).$$

Now ν_n and μ_n are both moment constants and also ν_n (and consequently μ_n too) has a particular form for all sufficiently large n. But we have also the moment functions $K(z)$ and $T(z)$ corresponding to ν_n and μ_n, both of which are regular and bounded in the half plane, with $K(n) = \nu_n$ and $T(n) = \mu_n$. It therefore follows from a known result (see for example, Titchmarsh [5], § 5.81) that given, ν_n or μ_n for all large n, each is uniquely determined for all n.

Consequently we have

$$\nu_n = 1 + \frac{r'}{n+1,}, \quad r' = \nu_0 - 1, \quad \text{i.e.}\ \frac{\mu_{n+1}}{\mu_n} = \frac{n+1}{n+1+r'}, \quad \text{for all } n.$$

Therefore

$$\mu_{n+1} = \frac{(n+1)\,(n)\,(n-1)\ldots 1}{(n+1+r')\,(n+r')\ldots(1+r')}\,\mu_0$$
$$= \mu_0 \frac{(n+1)\,!\,r'\,!}{(n+1+r')\,!}$$

and therefore $\mu_n = \mu_0 \cdot \binom{n + r'}{r'}^{-1} = \mu_0\,\bar{\mu}_n$ say, where $\bar{\mu}_n$ is the 'μ_n' of the (C, r') transformation.

Thus μ_n is the 'μ_n' of the (C, r') transformation, if the transformation (H, μ_n) is regular, or it is scalar multiple of the 'μ_n' of the (C, r') transformation.

The proof of the theorem is now complete from the fact that the method (C, r') is totally translative.

In conclusion I wish to express my thanks to Professor C. T. Rajagopal and Professor V. Ganapathy Iyer for their help and guidance. I am greatly indebted to Dr. B. Kuttner for his kindness in going through the original draft of this note, and for offering me helpful suggestions.

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ON THE EXISTENCE OF A NORM WEAKER THAN A GIVEN FAMILY OF NORMS ON A VECTOR SPACE

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1. LET $N_i(x)$ be a sequence of norms defined over a vector space E. Let us consider all those linear topologies T over E which are weaker than $N_i(x)$ for all i. (For brevity $N_i(x)$ is used to denote the norm as well as the corresponding topology). These include the lattice product tolopogy [3] of the $N_i(x)$, i.e. the strongest topology weaker than each of the $N_i(x)$ topologies. It is known that this topology is metrisable by the distance function $d(x, y)$ given by

$$d(x, y) = \sum_i \frac{1}{2^i} \frac{N_i(x-y)}{1+N_i(x-y)}.$$

We consider here the question when the lattice product topology is normable, and more generally when there exists a normable member of T, i.e. a topology which is normable and weaker than all the N_i. A necessary and sufficient criterion for this is given below. The interesting fact is also shown that there exist spaces E on which a family of norms can be imposed in such a way that there does not exist a normable topology weaker than this family.

THEOREM 1. *Let $N_i(x)$ be a sequence of norms over a space E. A necessary and sufficient condition for the existence of a norm topology on E weaker than each $N_i(x)$ is the existence of a sequence of positive numbers a_i such that $\Sigma\, a_i N_i(x)$ is convergent for every element x of E.*

PROOF. We note the well-known result that a norm $N(x)$ is topologically weaker than another norm $N_1(x)$ if and only if $N_1(x) \leqslant k\, N(x)$ for some constant k. Hence if $N(x)$ is a norm weaker than each $N_i(x)$, it is clear that $N_i(x)/N(x)$ is bounded above for all x in E and each i. Let $b_i = \text{l. u. b. } N_i(x)/N(x)$, $x \in E$, $a_i = 1/(2^i b_i)$.

Then the series $\Sigma\, a_i\, N_i\,(x)$ is convergent for each x, for,

$$\sum_i a_i\, N_i(x) = N(x) \sum a_i\, N_i(x)/N(x) \leqslant N(x) \sum a_i\, b_i = N(x). \quad (1.2)$$

Hence the condition is necessary. It is also sufficient since $\Sigma\, a_i N_i(x)$ itself serves as the desired norm; calling this $M(x)$, we have for each i,

$$M(x) > a_i\, N_i(x) \quad (1.3)$$

showing that $M(x)$ is weaker than each $N_i(x)$.

NOTE. The hypothesis on $N_i(x)$ in Theorem 1 can be easily seen to be equivalent to " For each element x of E, $N_i(x)$, as i varies, should be bounded above", which can be taken as an alternate hypothesis.

THEOREM 2. *If the lattice product topology of a sequence of topologies defined by norms $N_i(x)$ over a space E is normable by the norm $N(x)$, then there exist positive integers (a_i) such that $\Sigma\, a_i N_i(x)$ defines a norm equivalent to $N(x)$.*

PROOF. Consider the norm $M(x)$ defined by $M(x) = \Sigma\, a_i\, N_i\,(x)$, a_i being as in (1.1) of our Theorem 1. By (1.2) $M(x) \leqslant N(x)$ showing that $M(x)$ is topologically stronger than $N(x)$. But, by (1.3) it is weaker topologically than each $N_i(x)$ and hence weaker than $N(x)$. Hence $N(x)$ and $M(x)$ are topologically equivalent.

Theorem 1 can also be stated in the following equivalent form :

THEOREM 3. *If a sequence of norms $N_i(x)$ over a space E is such that for any given sequence of positive numbers (c_i) there is an element x such that $N_i(x) \geqslant c_i$, $(i = 1, 2, \ldots)$, then the lattice product of the $N_i(x)$ topologies is non-normable, and in fact there is no norm over E topologically weaker than all the N_i's.*

NOTE. The theorem does not preclude the possibility of defining a norm over a subset E_1 of E weaker topologically than all the N_i's.

2. In the following two sections we consider the application of these results.

THEOREM 4. *Let E be the space of all integral functions*

$$\alpha(z) = \sum a_n\, z^n$$

on which is imposed the family of norms

$$N[\alpha\ ;\ R\ ] = \sum |a_n| R^n,\ 0 < R < \infty.$$

Then there is no normable topology over E weaker than each $N(\alpha\,;R)$.

PROOF. The lattice product topology of $N(\alpha\,;R)$, $0 < R < \infty$, is the same as that of $N(\alpha\ ;R)$, $R = 1, 2, \ldots$. The result now follows from Theorem 3 on using the classical result on integral functions that, given any sequence of positive numbers $a_1, a_2, \ldots$ there is an integral function $\alpha(z)$ which takes values (a_n) at $z = 1, 2, \ldots$.

As a particular case, it follows from this that the lattice product of $N(\alpha\,;R)$, $0 < R < \infty$ is non-normable, a result proved in [1].

3. The space $\Gamma(R-0)$. On E the space of integral functions $\alpha = \Sigma\, a_n z^n$ let us consider the family of norms $N(\alpha\ ;r) = \sum\limits_{n=1}^{\infty} |a_n| r^n$. Let the resulting topological space be denoted by $\Gamma(r)$. As r increases the topology becomes weaker. The lattice product of the topologies as r varies in the range $0 < r < R$ (R fixed, > 0) is denoted by $\Gamma(R-0)$. This is metrisable by the distance function given by

$$D(\alpha,\,\beta) = \sum \frac{1}{2^n}\,\frac{N(\alpha-\beta\,;t_n)}{1+N\,(\alpha-\beta\,;t_n)}\ ,\ t_n = R-\frac{1}{n}.$$

THEOREM 5. *The completion of $\Gamma(R-0)$ is the space $\Gamma_1(R-0)$ or all power series $\alpha(z) = \Sigma\, a_n z^n$ with radius of convergence $\geqslant R$ metrized by the distance function $D(\alpha,\beta)$.*

As in the previous case we mean by this that under the distance function $D(\alpha,\beta)$, $\Gamma_1(R-0)$ is a complete metric space having $\Gamma(R-0)$ as a dense sub-set.

PROOF. First it is obvious that for all α, β in $\Gamma_1(R-0)$, the function $D\,(\alpha,\beta)$ is defined. Suppose now that (α_p) is a Cauchy sequence in the above metric, then

$$D(\alpha_p,\alpha_q) = \sum_{n=1}^{\infty} \frac{1}{2^n}\,\frac{N(\alpha_p-\alpha_q\,;t_n)}{1+N(\alpha_p-\alpha_q\,;t_n)} \leqslant \epsilon \text{ for } p, q \geqslant p_0.$$

Hence

$$N(\alpha_p - \alpha_q; t_n) = \sum_{s=1}^{\infty} | a_{p_s} - a_{q_s} | t_n^s$$
$$\leqslant \epsilon_n \text{ for all } n = 1, 2, \ldots \text{ and } p, q \geqslant p_0, \qquad (1)$$

where

$$\alpha_p = \sum_{s=1}^{\infty} a_{p_s} Z^s \text{ and } \epsilon_n = 2^n \epsilon/(1 - 2^n \epsilon).$$

Hence for each fixed s and each n ($n = 1, 2, 3, \ldots$), $|a_{p_s} - a_{q_s}| \leqslant \epsilon_n/t_n^s = \eta_n$ for $p, q \geqslant p_0$ where η_n, for fixed n, can be made as small as we like. Thus the sequence a_{p_s}(s fixed), $p = 1, 2, \ldots$, is convergent having a limit say a_s. Now making $q \to \infty$ in (1) we get

$$\sum_{s=1}^{\infty} | a_{p_s} - a_s | t_n^s \leqslant \epsilon_n \text{ for each } n \text{ and each } p.$$

Thus $\Sigma \, |a_s| \, t_n^s < \infty$ for each n; and $\alpha = \Sigma \, a_n z^n$ belongs to $\Gamma_1(R - 0)$.

We will now show that α is the limit of sequence α_p.

$$D(\alpha_p, \alpha) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{N(\alpha_p - \alpha; t_n)}{1 + N(\alpha_p - \alpha; t_n)}.$$

Given $\epsilon > 0$ choose N so large that $1/2^N < \epsilon/2$. Then in the above summation

$$\sum_{N+1}^{\infty} < \sum_{N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \frac{\epsilon}{2},$$

while

$$\sum_{1}^{N} < \sum_{1}^{N} \frac{1}{2^n} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2} \text{ for } p \geqslant p_0.$$

Hence

$$D(\alpha_p, \alpha) < \epsilon.$$

Again each element $\alpha = \Sigma a_n z^n$ with $\Sigma \, |a_n| \, t^n$ convergent for $t < R - 1/p$; $p = 1, 2, \ldots$, is the limit of the sequence

$$\alpha_n = \sum_{p=1}^{p} a_p z^p \quad \text{in } \Gamma(R-0)$$

showing that $\Gamma(R-0)$ is dense in $\Gamma_1(R-0)$.

4. The functionals of $\Gamma(R)$ and $\Gamma(R-0)$. It is known [2] that every functional f of $\Gamma(R)$ is of the form

$$f(\alpha) = \sum c_n a_n, \text{ where } \alpha = \sum a_n z^n \text{ and } |c_n|/R^n \text{ is bounded.}$$

By a known lemma†, it then follows that the functionals of $\Gamma(R-0)$ are of the same form with $|c_n|/r^n < M < \infty$ for some $r < R$. It should be noted that convergence in $\Gamma(R)$ is uniform convergence in the circle $|z| \leqslant R$, while that in $\Gamma(R-0)$ is uniform convergence over every circle contained in $|z| < R$.

THEOREM 6. *The space $\Gamma_1(R-0)$ is non-normable. In fact on $\Gamma_1(R-0)$ there is no norm weaker (topologically) than the family of norms*

$$N(\alpha; r),\ 0 < r < R.$$

PROOF. The proof is by an easy application of Theorem 3. We have only to produce for every sequence of positive numbers $a_n \to \infty$ a power series $\alpha(z)$ such that $\Sigma\, a_i N(\alpha;\ R-1/i)$ is not convergent.

Given the sequence $a_1, a_2, \ldots > 0$, construct an integral function $\beta(z)$ such that $\beta(z) = 1/a_i$ for $z = i$ $(i = 1, 2, \ldots)$.

Let

$$\alpha(z) = \beta\left(\frac{1}{R-z}\right).$$

Then

$$\alpha(z) \in \Gamma_1(R-0).$$

† Lemma (A). Let (T_i) be a family of comparable linear topological spaces over a vector space E and T their Lattice product; T_i^*, T^* denote the set of linear continuous functionals in T_i, T respectively, then

$$T^* = \bigcup_i T_i^*,$$

In other words $f \in T^*$ if and only if $f \in T_i^*$, for some i.

Now

$$N(\alpha; R - 1/n) \geqslant \text{u. b.} \, |\alpha(z)|, \text{ on the circle } |z| = R - 1/n,$$
$$\geqslant |\alpha(R - 1/n)| = \beta_i = 1/a_i.$$

Thus

$$\sum a_i N(\alpha ; R - 1/i) \geqslant \sum a_i \frac{1}{a_i} = \infty$$

which establishes the result.

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A PROPERTY OF TRANSFORMATIONS OVER A SEQUENCE OF SPACES

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1. The object of this paper is to show that a theorem on linear continuous transformations on the space of integral functions proved by V. Ganapathy Iyer [1] holds substantially for general normed vector spaces, and to obtain an extension of the result for transformations from a directed family topologies directed by the relation "weaker than" defined on a vector space E to a family of locally bounded topologies defined over another vector space V.

2. Let E be a vector space on which a sequence of norms $N_i(x)$, $i = 1, 2, \ldots$ are imposed. Let $N_i(E)$ denote the space E topologized by the norm $N_i(x)$, and $M(E)$ the space E topologized by the lattice product topology [2] of the sequence of norm topologies N_i. It is well known that $M(E)$ is metrizable with the metric $D(x, y)$ given by

$$D(x, y) = \sum \frac{1}{2^i} \frac{N_i(x-y)}{1 + N_i(x-y)} \cdot$$

Further let $N_i(x)$ be an increasing function of i so that $N_i(x) \leqslant N_i(x)$, $i < j$ so that topology of $N_j(E)$ is weaker than that of $N_i(E)$.

Using the notation of [1], let $N(i \to j)$ denote a linear continuous transformation whose domain is $N_i(E)$ and the range is in $N_i(E)$. Let the family of all such transformations be denoted by $F(i \to j)$. Also let $T(\infty \to \infty)$ denote a linear continuous transformation of $M(E)$ into itself, and let the family of all such transformations be denoted by $F(\infty \to \infty)$. Then every $T(\infty \to \infty)$ can be specified in terms of a suitable family $F(i \to j)$ by means of

Theorem 1. *A transformation T belongs to $F(\infty \to \infty)$ if and only if to each $j > 0$ there is an $i > 0$ such that*

$$N_j\,[\,T(x)] \leqslant K_{ij}\,N_i(x), \qquad x \in E,$$

K_{ij} being a constant (finite).

It is easy to see that this result is equivalent to

THEOREM 1a. $F(\infty \to \infty) = \prod_{j>0} [\sum_i F(i \to j)]$.

In other words, a linear continuous transformation T from E into E is continuous in the lattice product topology if and only if for every j it is continuous from $N_i(E)$ to $N_j(E)$ for a suitable i.

Ganpathy Iyer's result is obviously a special case of this. To prove Theorem 1*a* we require the following

LEMMA 1. *Let S be a linear subspace of E and $x_0 \in E$. If for all $y \in S$, $D(x_0, y) \geqslant \delta > 0$, then $N_i(x_0 - y) > \delta_1 = \delta/(2-\delta)$ for all $i \geqslant i_0$ given by*

$$2^{i_0-1} > \frac{1}{\delta}\,.$$

PROOF. Using the condition on i_0, we have

$$\sum_{i_0+1}^{\infty} \frac{1}{2^i}\,\frac{N_i\,(x_0-y)}{1+N_i\,(x_0-y)} < \sum_{i_0+1}^{\infty} \frac{1}{2^i} < \frac{\delta}{2}$$

This result, combined with the hypothesis $D(x_0, y) \geqslant \delta$ gives

$$\sum_{1}^{i_0} \frac{1}{2^i}\,\frac{N_i\,(x_0-y)}{1+N_i\,(x_0-y)} > \frac{\delta}{2}\,,$$

or, since $N_i(x)$ monotonically increases with i,

$$\frac{N_{i_0}\,(x_0-y)}{1+N_{i_0}\,(x_0-y)} \sum_{1}^{i_0} \frac{1}{2^i} > \frac{\delta}{2}$$

so that $N_{i_0}(x_0 - y)\,\delta(2-\delta) = \delta_1 < 0$ since $0 < \delta < 1$, and the lemma follows.

PROOF OF THEOREM 1*a*. Since a continuous transformation has the property that it continues to be continous if the topology of the range is strengthened or that of its domain weakened, it follows that a $T(\infty \to \infty)$ is a $T(\infty \to j)$ for every $j > 0$. Hence

$$F(\infty \to \infty) \subset \prod_{j>0} F(\infty \to j). \tag{1}$$

Conversely to show that

$$F(\infty \to \infty) \supset \prod_{j>0} F(\infty \to j) \tag{2}$$

suppose that a transformation T in $M(E)$ is discontinuous; then there exists a sequence (x_p) of elements of E such that

$$D(0, x_p) \to 0 \text{ as } p \to \infty, \text{ but } D[T(x_p), 0] \geqslant \delta > 0 \quad p = 1, 2, \ldots.$$

By the Lemma, $N_j[\,T(x_p)\,] \geqslant \delta_1 = \delta/(2-\delta)$. for $j \geqslant j_0$, where j is determined by $1/2^{j_0} \leqslant \dfrac{\delta}{2}$. Hence T is not in $F(\infty \to \infty)$ for $j \geqslant j_0$, and hence (2) follows. Combining (1) and (2) we have

$$F(\infty \to \infty) = \prod_{j>0} F(\infty \to j)\,. \tag{3}$$

Next let $j > 0$ be fixed, then any $T(i \to j)$ is a $T(\infty \to j)$ and so

$$\sum_{i>0} F(i \to j) \subset F(\infty \to j).$$

For the reverse implication, suppose that a $T(i \to j)$ is not continuous for every $i > 0$. Then for each integer $p > 0$ there is an element (x_p) in $N_p(E)$ such that $N(x_p) \leqslant 1/p$ while $N_j[T.(x_p)] \geqslant 1$.

Since N_p increases with p, the sequence $(x_p) \to 0$ in every norm N_i, for, if we take the norm N_q(q fixed), we have

$$N_q(x_q) \leqslant \frac{1}{q},$$

$$N_q(x_{q+1}) \leqslant N_{q+1}(x^{q+1}) \leqslant \frac{1}{q+1},$$

and in general,

$$N_q(x_{q+r}) \leqslant \frac{1}{q+r}, \quad r = 1, 2, 3, \ldots$$

Hence the sequence (x_p), $p = 1, 2, 3, \ldots, \to 0$ in every norm N_q, $q = 1, 2, \ldots$ and hence $D(x_p, 0) \to 0$ as $p \to \infty$. But since by assumption $N_j(T(x_p)) \geqslant 1$ for every p, it follows that such a T is not a T $(\infty \to j)$; thus

$$\sum_{i>0} F(i \to j) \supset F(\infty \to j). \tag{4}$$

Combining (3) and (4) we have the required result.

3. We will now prove the following more general

THEOREM 2. *Let τ_α be a family of topological vector spaces on the same base space E, directed by the relation "topologically weaker," i.e. such that given any two and hence any finite number of members of the family, there will be one member of the family which will be weaker than all of them. Let Γ_β be a family of locally bounded (or as a special case normed) topologies on another vector space V. Let τ, Γ denote the lattice products of these families in order. Then a linear map f from (E, τ), i.e. the space E with the topology τ, to (V, Γ) is continuous if and only if for every Γ_β there exists a τ_α such that f is continuous from (E, τ_α) to (V, Γ_β).*

In other words, denoting by $L(\alpha, \beta)$ the totality of linear continuous maps from (E, τ_α) to (V, Γ_β) we have.

$$L(\tau, \Gamma) = \prod_\beta \Big(\sum_\alpha L(\alpha, \beta) \Big),$$

the left side denoting the totality of linear continuous maps from (E, τ) to (V, Γ). This is an immediate consequence of the following two lemmas.

LEMMA 2. *A map f from a topological space E to the lattice product Γ of a family of topologies Γ_β on any set V (E and V need not be vector spaces nor need f be a linear map) is continuous if and only if it is continuous from E to (V, Γ_β) for every β.*

PROOF. The "only if" part is obvious since the continuity of a map is preserved by strengthening the topology of the range. To

prove the "if" part let U be a neighbourhood of $f(x)$, $x \in E$, in the Γ topology. Thus U may be taken to be $U_1 \wedge U_2 \wedge \ldots U_n$, ($n$ finite, and U_i is a neighbourhood of $f(x)$ in Γ_{β_i} topology). Then the continuity of f with respect to the Γ_{β_i} topology shows that $f^{-1}(U_i)$ must contain a neighbourhood $W_i(x)$ of x in E. Thus, $f^{-1}(U_1 \wedge U_2 \wedge \ldots U_n) = f^{-1}(U_1) \wedge f^{-1}(U)_2 \wedge \ldots f^{-1}(U_n)$ contains the neighbourhood $W_1(x) \wedge W_2(x) \wedge \ldots W_n(x)$ of x; this shows that f is continuous in the Γ topology also, thus establishing the lemma.

LEMMA 3. *With the same notation as in Theorem 2, a linear map f is continuous from (E, τ) to (V, Γ_β) (for a fixed β) if and only if it is continuous from (E, τ_α) to (V, Γ_β) for some α.*

PROOF. First we note that continuity at the origin implies continuity everywhere. Also since Γ_β is locally bounded, a complete system of neighbourhoods of the origin in Γ_β is of the form kU, where k takes all values in the range $0 < K < \infty$, and U is a fixed neighbourhood. Thus f is continuous from (E, τ) to (V, Γ_β) if and only if $f^{-1}(KU) = Kf^{-1}(U)$ is a neighbourhood of the origin in τ, i.e. if and only if $f^{-1}(U) \supset W_1 \wedge W_2 \wedge \ldots W_r$, r finite, where W_i is a neighbourhood in the τ_{α_i} topology. But this is equivalent to saying that $f^{-1}(U) \supset W_\alpha$, where W_α is a neighbourhood of the origin in some topology τ_α weaker than $\tau_{\alpha_1} \ldots \tau_{\alpha_r}$, (and such a topology exists in the family by hypothesis). Thus, by arguing again as above, this means that f is continuous from the τ_α topology to the Γ_β topology. This establishes the lemma.

The proof of Theorem 2 now follows easily, for, f is continuous from (E, τ) to (V, Γ) if and only if for each β, f is continuous from (E, τ) to (V, Γ_β) (Lemma 2), and this, by Lemma 3, is true if and only if f is continuous for each (V, Γ_β) and some (E, τ_α) (which might depend on β), and this is what we set out to prove. Theorem 1a is a particular case of this with E and V coinciding, and also the families τ_α, Γ_β which are there comparable monotonic normed topologies.

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NOTE ON A THEOREM OF MAZUR AND ORLICZ IN SUMMABILITY

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IN this note we shall prove some extensions of certain theorems of Mazur and Orlicz [5] and some applications thereof. In these theorems a sequence $x = \{x_n\}$ is transformed into another sequence $y = \{y_n\}$ by the relations

$$y_i \equiv A_i(x) = \sum_{n=1}^{\infty} a_{in} x_n \quad (i = 1, 2, 3, \ldots), \tag{1}$$

where the numbers (a_{in}) are the elements of a matrix $A \equiv (a_{in})$ of a specified type.

If $\lim_{n\to\infty} y_n$ exists and is finite, we say that the sequence $x = \{x_n\}$ is *A-summable* to the value of the limit, and we write

$$A\text{-}\lim x_n = \lim_{n\to\infty} y_n.$$

If $A\text{-}\lim x_n = \rho \lim_{n\to\infty} x_n$ (where ρ is any fixed number) whenever $\lim_{n\to\infty} x_n$ exists, then we say that the matrix or method A is *multiplicative* ρ, and call the method *permanent* (generally) in the case $\rho = 1$ and *permanent for null sequences* if $\lim_{n\to\infty} x_n = 0$ implies $A\text{-}\lim x_n = 0$.

Denoting by (c), (c_0) the sets of convergent sequences, and sequences converging to 0 respectively, we say, in relation to a method A, that $x \in (A)$ or $x \in (A_0)$ according as $Ax = y \in (c)$ or $Ax = y \in (c_0)$. We denote the set of bounded sequences by (m).

The method A is said to be *convergence-preserving* if $(c) \subset (A)$, i.e. if the existence of $\lim_{i\to\infty} x_n$ implies that of $A\text{-}\lim x_n$. Similarly,

the method A is said to be *convergence-preserving for null sequences* if $(c_0) \subset (A)$. We use also the following notation. Let $\beta = \{\beta_n\}$ be any sequence of numbers. If $\sum_{n=1}^{\infty} \beta_n x_n$ is convergent and A-lim x_n exists and is equal to $\sum_{n=1}^{\infty} \beta_n x_n$, then we say that $x \in (A_\beta)$. Thus (A_0) will be the set of all sequences $x = \{x_n\}$ such that $A\text{-lim}\, x_n = 0$, consistent with our previous notation. If the method A is convergence-preserving for null sequences, then it is known that $\alpha_n = \lim_{i \to \infty} a_{in}$ *necessarily exists for each* $n = 1, 2, 3, \ldots$ and also that $\sum_{n=1}^{\infty} |\alpha_n| < \infty$. We shall find that the set (A_α) is closely connected with the summability properties of the method A.

Lastly, we say that the matrix or method A is a *U-method* if, y_i $(i = 1, 2, 3, \ldots)$ being defined by (1), the system of equations $y_i = 0$ $(i = 1, 2, 3, \ldots)$ has only one solution, viz. $x_n = 0$ $(n = 1, 2, 3, \ldots)$. If A has the above property under the additional hypothesis that $x = \{x_n\} \in (m)$, then A is called a *U-method for bounded sequences.*

Many of the results proved by Mazur and Orlicz [5] for a method which is permanent for null sequences can be extended, without any significant change in the proofs, to a method A which is convergence-preserving for null sequences or is convergence-preserving. We shall see that the separability of certain subsets of $(A)(m)$, where $(A)(m)$ denotes the set of bounded sequences summable by A, is equivalent to their images by A being closed under the usual norm in (m). We obtain also extensions of two theorems due respectively to Copping [2] and the author [6]. Theorem 7 gives a set of sufficient conditions for a method A to sum some bounded divergent sequences, and we show by means of examples that these conditions are in a sense the best possible.

THEOREM A. *If the method A is permanent for null sequences, and if there are bounded divergent sequences A-summable to 0, then*

$(A_0)(m)$, i.e. the set of bounded sequences A-summable to 0, is non-separable in (m) under the norm $\|x\| = \sup_n |x_n|$.†

THEOREM B. (Agnew [1, p. 97]). *Let A be multiplicative zero. Then $(A_0)(m)$ is non-separable in (m).*

THEOREM 1. *Let A be convergence-preserving for null sequences. Then $(A_\alpha)(m)$ is separable in (m) if and only if $(A_\alpha)(m) = (c_0)$.*

In particular, if A is permanent for null sequences, then $(A_0)(m)$ is separable in (m) if and only if $(A_0)(m) = (c_0)$.

PROOF. To prove the theorem it is enough to show that if $(A_\alpha)(m)$ is separable, then $(A_\alpha)(m) = (c_0)$, the converse statement being obvious. By hypothesis, $\sup_n \sum_{n=1}^{\infty} |a_{in}| < \infty$ and $\alpha_n = \lim_{i\to\infty} a_{in}$ exists for each $n = 1, 2, 3, \ldots$. Hence the series $\sum_{n=1}^{\infty} \alpha_n$ is absolutely convergent, as also is the series $\sum_{n=1}^{\infty} \alpha_n x_n$ for every $x = \{x_n\} \in (m)$. Now let

$B \equiv (b_{in})$, where $b_{in} = a_{in} - \alpha_n$ $(i, n = 1, 2, 3, \ldots)$; then $(B)(m) = (A)(m)$ and $(B_0)(m) = (A_\alpha)(m)$. Also, the method B is permanent for null sequences, for we have

$$\sum_{n=1}^{\infty} |b_{in}| = \sum_{n=1}^{\infty} |a_{in} - \alpha_n| \leqslant \sum_{n=1}^{\infty} |a_{in}| + \sum_{n=1}^{\infty} |\alpha_n| < H < \infty,$$

where H is independent of i; and $\lim_{i\to\infty} b_{in} = 0$ for each $n = 1, 2, 3, \ldots$. It follows now from Theorem A that if the set $(B_0)(m)$ is separable in (m), then $(B_0)(m) \subset (c)$.

Now if $(B_0)(m) \not\subset (c_0)$, it follows from the last remark that there must exist a sequence $x = \{x_n\} \in (c) - (c_0)$ such that $Bx \in (c_0)$. Let $\lim_{n\to\infty} x_n = l$, and let $\dot{l}$ denote the constant sequence $\{e\}$. Then $y = x - \dot{l} \in (c_0)$, and since B is permanent for null sequences we

†Mazur and Orlicz [5, § 3.6.2]; cf. [4], [1] where the result is given in a slightly less general form.

shall have that $By \in (c_0)$. But $Bx \in (c_0)$, and therefore $B(x-y) = Bl \in (c_0)$: i.e. $l \sum_{n=1}^{\infty} b_{in} \to 0$ as $i \to \infty$. Since $l \neq 0$, it follows that $\lim_{i\to\infty} \sum_{n=1}^{\infty} b_{in} = 0$, and thus B will satisfy all the conditions for being a convergence-preserving multiplicative zero method. Theorem B now demands that $(B_0)(m) = (A_\alpha)(m)$ be non-separable in (m), thus leading to a contradiction. Therefore $(B_0)(m) \subset (c_0)$; and since $(B_0)(m) \supset (c_0)$, we see that $(B_0)(m)$ is precisely the set (c_0).

THEOREM 2. *Let A be convergence-preserving. Then $(A)(m)$ is separable in (m) if and only if $(A)(m) = (c)$.*

PROOF. Since (c) is separable, we have to prove only that the condition is necessary. Now let the matrix B be defined as in the proof of Theorem 1. Then it is easily seen that B is a convergence-preserving method which is permanent for null sequences, and that $(A_\alpha)(m) = (B_0)(m) \subset (B)(m) = (A)(m)$. Therefore, if $(A)(m)$ is separable and therefore also its subset $(A_\alpha)(m) = (B_0)(m)$, it follows from the proof of Theorem 1 that $(B_0)(m) = (A_\alpha)(m) = (c_0)$.

If $(A)(m) = (B)(m) \not\subset (c)$, there must exist then a bounded divergent sequence $x = \{x_n\}$ such that A-lim x_n exists. Let B-lim $x_n = l$, and let

$$\chi(A) \equiv \lim_{i\to\infty} \sum_{n=1}^{\infty} a_{in} - \sum_{n=1}^{\infty} \alpha_n = \lim_{i\to\infty} \sum_{n=1}^{\infty} b_{in}.$$

Now if $\chi(A) = 0$, then B will satisfy all the conditions for being a multiplicative zero matrix and we shall have $(B_0)(m) \supset (c)$. This contradicts that $(B_0)(m) = (c_0)$. Therefore $\chi(A) \neq 0$.

Let $u = \{u_n\}$, $u_n = x_n - l/\chi(A)$. Then B-lim $u_n = B$-lim $x_n - l = 0$. But u is a bounded divergent sequence, and the conclusion $Bu \in (c_0)$ contradicts the fact $(B_0)(m) = (c_0)$. Thus we see that $(B)(m) \subset (c)$, and since $(A)(m) = (B)(m)$ and A is convergence-preserving, it follows that $(A)(m) = (c)$. This proves the theorem.

THEOREM 3. *Let A be convergence-preserving, and let*

$$\chi(A) \equiv \lim_{i\to\infty} \sum_{n=1}^{\infty} a_{in} - \sum_{n=1}^{\infty} \alpha_n = 0.$$

Then there exists a bounded divergent sequence $t = \{t_n\}$ summable by A to the value $\sum_{n=1}^{\infty} \alpha_n t_n$.

PROOF. Let B be defined as in the proof of Theorem 1. Then B is multiplicative zero, and by Theorem B we shall have that (B_0) (m) is not separable. But (B_0) $(m) = (A_\alpha)$ (m), and it follows from the proof of Theorem 1 that (A_α) (m) contains a (bounded) divergent sequence.

COROLLARY. *Let A be convergence-preserving. If $\chi(A) = 0$, then A sums some bounded divergent sequences as well as some unbounded sequences. If $\chi(A) \neq 0$, then the existence of a bounded divergent sequence summable by A is sufficient but not necessary in order that A might sum some unbounded sequences also.*

The truth of the statement in the case $\chi(A) = 0$ and the sufficiency part in the case $\chi(A) \neq 0$ are evident from Theorem 3 and the fact that $(A) \subset (m)$ implies $(A) = (c)$, (Zeller [8], Satz 7.1). The remaining final part of the corollary is seen to be true by taking, for example, A to be the matrix (a_{in}), where $a_{ii} = 1, a_{i,i+1} = -\frac{1}{2}$, $a_{in} = 0$ if $n \neq i$ or $i + 1$, for all $i = 1, 2, 3, \ldots$. Then (A) $(m) \subset (c)$, but $(A) \not\subset (m)$, for $\{2^n\} \in (A_0)$.

Let p be any positive integer, and let (m_p) denote the subset of (m) each of whose elements is a sequence whose first p terms at least are all zero; i.e. $x = \{x_n\} \in (m_p)$ if and only if $x \in (m)$ and $x_n = 0$ for $n \leqslant p$. Let us consider now the set $R_{(A)(m_p)}(A)$ consisting of all y such that $y = Ax$, where $x \in (A)(m_p)$. If $Ax = 0$ has no bounded divergent solution, then by a theorem of Mazur and Orlicz ([5], § 3.7), the equation has got a finite number of linearly independent solutions in (c_0). It follows that we can choose the integer p so large that $x = 0$ is the only solution in (m_p) of $Ax = 0$.

Then A will map $(A)\,(m_p)$ onto $R_{(A)(m_p)}(A)$ in a one-to-one manner. Further, $(A)\,(m_p)$ is obviously a closed subspace of $(A)\,(m)$, and it is known that $(A)\,(m)$ is complete. It follows that $(A)\,(m_p)$ is a Banach space, where the norm is defined as usual by $\|x\| = \sup_n |x_n|$, $x = \{x_n\}$.

Therefore, if $R_{(A)(m_p)}\,(A) \subset (c)$ is closed (and therefore complete) under the norm as defined above, then, in view of the one-to-one mapping, we see that $(A)\,(m_p)$ is a Banach space also when the norm is defined as $\|x\|_1 = \|Ax\|$. Consequently there exists a number $k > 0$ such that $\|x\| \leqslant k\,\|Ax\|$.

We have always $\|Ax\| \leqslant \|A\|.\ \|x\|$. Hence, in the converse case, if there exists a constant $k > 0$ such that $\|x\| \leqslant k\,\|Ax\|$, it follows that the norms $\|x\|$ and $\|x\|_1$ are equivalent for $(A)(m_p)$, and therefore $R_{(A)(m_p)}\,(A)$ is complete under the usual norm.

The results of §§ 3.7.1 and 3.7.2 of Mazur and Orlicz [5] can now be summed up, and stated more generally, as follows.

THEOREM 4. *Let A be convergence-preserving. Then in order that $(A)(m) \subset (c)$, it is necessary and sufficient that $R_{(A)(m_p)}\,(A)$ be complete under the metric $\|x\| = \sup_n |x_n|$ for some integer p.*

If A is a U-method for bounded sequences, then p may be taken to be zero, where $(m_0) = (m)$.

PROOF. By § 3.7.2 of Mazur and Orlicz's paper [5], by a trivial extension of their result from permanent to convergence-preserving methods, we shall have that $(A)(m) \subset (c)$ if and only if there exists a number $k > 0$ and an integer p such that for every system $\lambda_1, \lambda_2, \ldots, \lambda_r$ of numbers, the inequality

$$\sup_i \left| \sum_{n=1}^{r} a_{i,n+p}\,\lambda_n \right| \geqslant k \sup |\lambda_n|$$

is true.

Now the above inequality is seen to be equivalent to the inequality

$$\sup_i \left| \sum_{n=1}^{\infty} a_{i,n+p}\,\lambda_n \right| \geqslant k \sup_n |\lambda_n|,$$

where $\{\lambda_n\} \in (m)$, and this can be written as

$$\|Ax\| \geqslant k \|x\|, \text{ where } x \in (m_p).$$

Since this is a necessary and sufficient condition for $(A)(m_p)$ to be complete under $\|x\|$, the theorem is proved.

We now give an application of the above ideas to two problems concerning reciprocals of matrices. Let I be the matrix of the identity transformation. If $\|A\| \equiv \sup_i \sum_{n=1}^{\infty} |a_{in}| < \infty$, $\|B\| \equiv \sup_i \sum_{n=1}^{\infty} |b_{in}| < \infty$, and $BA = I$, then B is not only a left-hand reciprocal of A, but also the inverse of A for bounded sequences; that is, $x \in (m)$ and $y = Ax$ imply that $x = By$. Thus A is a U-method for bounded sequences. Also, the inequality

$$\|x\|_1 \geqslant k \|x\|$$

holds for all $x \in (m)$, where we may take $k = \|B\|$.

Hence $(A)(m)$ and therefore also its subsets $(A_\alpha)(m)$ and $(A_0)(m)$ are separable under the norm $\|x\|$. Theorems 1 and 2 now yield the following result, part of which has been proved by Copping [2].

THEOREM 5. *Let the method A be convergence-preserving, or permanent for null sequences, respectively. Let the matrix B be such that $\|B\| < \infty$ and $BA = I$. Then $(A)(m) = (c)$, $(A_\alpha)(m) = (c_0)$ respectively.*

We may use Theorem 5 to supplement a result which I proved earlier [6] on more or less the same lines. We have

THEOREM 6. *Let the method A be convergence-preserving, or permanent for null sequences, respectively. Let the matrix B satisfy the conditions $\|B\| < \infty$, $BA = AB = I$. Then B is convergence-preserving, or permanent for null sequences, respectively.*

PROOF. Since $AB = I$, we have $x = (AB)x = A(Bx)$ for all $x \in (m)$. Since $BA = I$ and $\|B\| < \infty$, it follows from Theorem 5 that $x \in (c)$, or $x \in (c_0)$, respectively, implies that $Bx \in (c)$, or $Bx \in (c_0)$, respectively. This proves the theorem.

We may remark that it is not enough to assume that the method in Theorem 6 is merely convergence-preserving for null sequences. For, let the matrix $A = (a_{in})$ be defined by $a_{i1} = 1$ for $i \geqslant 1$, $a_{ii} = (-1)^i$ for $i \geqslant 2$, and $a_{in} = 0$ whenever n is neither unity nor equal to i. Then the matrix $B = (b_{nk})$, where $b_{n1} = (-1)^{n+1}$ for $n \geqslant 1$, $b_{nn} = (-1)^n$ for $n \geqslant 2$, and $b_{nk} = 0$ whenever k is neither unity nor equal to n, satisfies the conditions $\| B \| < \infty$, $BA = AB = I$. But B is not convergence-preserving for null sequences, since $\lim_{n\to\infty} b_{n1}$ does not exist.

It is known that if a convergence-preserving method sums no unbounded sequences, then it is equivalent to convergence. We now seek conditions under which a convergence-preserving method which sums no bounded divergent sequences may be asserted to be equivalent to convergence. In other words, we seek results of the type where A-summability of an unbounded sequence is part of the hypothesis and the A-summability of a bounded divergent sequence is the conclusion. Theorem 3 and its corollary show that $\chi(A) = 0$ is a sufficient but not a necessary condition in order that $(A)(m) \neq (c)$ for a convergence-preserving method A. We now give two results involving reversible matrices of type M. We recall that the matrix A is said to be *reversible* if A is a U-method and $R_{(A)}(A) = (c)$. Also, the matrix A is said to be of *type M* if the conditions $\sum_{i=1}^{\infty} |\alpha_i| < \infty$, $\sum_{i=1}^{\infty} \alpha_i a_{in} = 0$ $(n = 1, 2, 3, \ldots)$ imply $\alpha_i = 0$ for all $i = 1, 2, 3, \ldots$.

THEOREM 7. *Let A be a reversible, convergence-preserving method of type M, and let $(A) \neq (c)$. Then $(A)(m) \neq (c)$. That is, if the reversible type M method A is strictly stronger than convergence, then there exist bounded divergent sequences which are summable by A.*

To prove this and the next theorem we shall need the following results which I have proved elsewhere [7]:

LEMMA 1. *Let A be a convergence-preserving method of type M with $\chi(A) \neq 0$. Then $R_{(c)}(A)$ is dense in (c).*

LEMMA 2. *Let A be a method, with* $\sup_i \sum_{n=1}^{\infty} |a_{in}| < \infty$, *and let* $R_{(m)}(A)$ *be dense in* (c_0). *Then A is of type M.*

PROOF OF THE THEOREM. We have $R_{(A)}(A) = (c)$, since A is reversible, and the mapping by A of (A) onto (c) is one-to-one. Also, A is certainly a U-method for bounded sequences. Now if $(A)(m) = (c)$, then $R_{(c)}(A)$ is complete in (c) under the usual norm, by Theorem 4. Now if $\chi(A) = 0$, then the result follows at once from Theorem 3. If $\chi(A) \neq 0$, then by Lemma 1 we have that $R_{(c)}(A)$ is dense in (c) under the same metric. It follows then that $R_{(c)}(A) = (c)$, since (c) is known to be complete under the metric. We are thus led to a contradiction, and therefore it follows that $(A)(m) \neq (c)$.

The above theorem is the best possible in the sense that while the five conditions (i) *A is convergence-preserving*, (ii) *A is reversible*, (iii) *A is of type M*, (iv) $(A)(m) \subset (c)$, and (v) $(A) \neq (c)$ form an inconsistent set, yet no four of them are inconsistent. To see the truth of this statement, we need only consider the following examples.

(i)$'$: Let A be the diagonal matrix with $a_{nn} = n$ $(n = 1, 2, \dots)$; this satisfies all the conditions except (i).

(ii)$'$: Let A be the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & . & . \\ 0 & 2 & 1 & 0 & 0 & . & . \\ 0 & 0 & 2 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

A is not reversible, since it transforms $\{(-2)^n\}$ into $\{0\}$. But it satisfies the remaining four conditions. It may be noted that A is a U-method for bounded sequences.

(iii)′ : Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . \\ 2 & 1 & 0 & 0 & 0 & . & . \\ 0 & 2 & 1 & 0 & 0 & . & . \\ 0 & 0 & 2 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Then A is seen to satisfy all the conditions except (iii).

(iv)′ : The $(C,1)$ matrix satisfies all the conditions except (iv).

(v)′ : Take A to be the identity matrix, for which the conditions (i)-(iv) are true, but not the condition (v).

NOTE : The example (iii)′ and Theorem 7 complete a result due to Hill [3] that a reversible permanent method A which is not of type M has $(A) \not\subset (m)$; i.e. $(A) \neq (c)$. We see now that it may happen that $(A)(m) = (c)$.

THEOREM 8. *Let A be a reversible convergence-preserving method with $(A) \neq (c)$, and let each column of the associated matrix B be bounded. Then $(A)(m) \neq (c)$.*

PROOF. We may assume that $\chi(A) \neq 0$; for if $\chi(A) = 0$, we get the result immediately from Theorem 3. Also A sums each sequence which forms a column of B, since in fact $AB = I$. Let Δ denote the set of sequences $\{\delta_n^i\}$, $i = 1, 2, 3, \ldots$, where for each i, $\delta_i^i = 1$ and $\delta_n^i = 0$ for $n \neq i$. Then $R_\Delta(B)$ consists of elements each of which is a column of the matrix B, and further $R_{R_\Delta(B)}(A) = \Delta$. But the set of all finite linear combinations of elements of Δ is dense in (c_0), and since $R_\Delta(B)$ is a subset of the vector space (m), it follows that $R_{(m)}(A)$ is dense in (c_0). From Lemma 2 it follows that A is of type M. The result now follows from Theorem 7.

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TWO TAUBERIAN THEOREMS FOR FUNCTIONS SUMMABLE (A)

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1. Introduction. Let $s \equiv s(u)$ be a function of bounded variation in every finite interval of $u \geqslant 0$ and supposed, for simplicity, to be such that $s(0) = 0$. Then for such a function $s(u)$ we define, as usual,

the Laplace transform :

$$L(t, s) = \int_0^\infty e^{-tu}\, d\{s(u)\},\ t > 0, \tag{1}$$

the Cesàro transform (C, α) of order $\alpha \geqslant 0$:

$$\left.\begin{aligned} C^\alpha(x, s) &= \frac{\alpha}{x^\alpha}\int_0^x (x-u)^{\alpha-1}\, s(u)\, du,\ \alpha > 0 \\ C^0(x, s) &= s(x). \end{aligned}\right\} \tag{2}$$

summability (A) to S : $L(t, s) \to S$ as $t \to +0$, (3)

summability (C, α) to S : $C^\alpha(x, s) \to S$ as $x \to +\infty$. (4)

The notation here is in conformity with that of Hardy ([2], pp. 110-2), so that $C^\alpha(x, s)$ here is the same as $C_\alpha(x)$ in a paper by Rajagopal [5] of which this note is a sequel. In this notation, a theorem of Rajagopal's ([4], Theorem 1) states that a function $s(u)$ summable (A) to S will be summable $(C, k+1)$ to S if $k \geqslant 0$ is such that

$$C^k(x, s) - C^{k+1}(x, s) = O_L(1),\ x \to \infty. \tag{5}$$

Another theorem of his ([5], Theorem Ā) extends this result to a class of integral transforms

$$\Phi(t, s) = \int_0^\infty \phi(tu)\, d\{s(u)\},\ t > 0,$$

where $\phi(u)$ satisfies suitable conditions and can be, in particular either the Lambert kernel or the Stieltjes kernel†; and the proof he has given of this extension follows the lines of a proof suggested by Chandrasekharan and Minakshisundaram ([1], proof of Theorem 3.83) for the case $k = 0$, $\phi(u) = e^{-u}$ due to O. Szasz ([6], Theorem 1 or [10], pp. 635-8). It is the object of this note to prove the following simple generalization of Rajagopal's theorem for the Laplace transform stated above.

THEOREM I. *If $s(u)$ is summable (A) to S and there are real numbers $k \geqslant 0$, c such that*

$$C^k(x, s) - (1 + c)\ C^{k+1}(x, s) = O_L(1),\ x \to \infty, \tag{6}$$

then $s(u)$ is summable $(C, k + 1)$ to S.

The proof of Theorem I is similar to that of an analogous theorem for a sequence summable (A), involving the Hölder transforms of the sequence of real orders k and $k + 1$ related as in (5), given by Jakimovski and myself ([3], Theorem I). There is an extension of Theorem I for the transform $\Phi(t, s)$. But this extension is not given here since its proof is exactly like that of Theorem I, but depending on the positive Tauberian theorem for the transform Φ ([5], Lemma 1 ; or [2], Theorem 236), a theorem for the product of transforms (C, α) and Φ ([5], Theorem II) and the analogue of Tauber's second theorem for the transform Φ (e.g. [5], Corollary A with $k = 0$) instead of depending on Lemmas 2, 3 and the case $c = 0$ of Lemma 4 of the next section.

† One consequence of this extension is that we can take $k=0$ in (5), particularise $\phi(u)$ successively as the Laplace-Abel and the Lambert kernel, and obtain, along with certain well-known Tauberian theorems for the convergence of series summable (A, λ) in Hardy's sense ([2], § 4.7), the exact counterparts of the theorems in [5], Corollary $\bar{A}$, for the convergence of series summable (L, λ) where the method of summability (L, λ) is the generalization of the Lambert or L method ([2], Appendix IV) exactly as the method (A, λ) is that of the method (A).

2. Preliminary results. The first two of the results given below are well known (e.g. [2], p. 112 and Theorem 98).

LEMMA 1. *If we write*

$$m(x) = C^1(x, s) = \frac{1}{x}\int_0^x s(u)\,du, \tag{7}$$

then, for any $k \geqslant 0$,

$$C^k(x, m) = \frac{1}{x}\int_0^x C^k(u, s)\,du, \tag{8}$$

$$C^{k+1}(x, s) = (k+1)\,C^k(x, m) - k\,C^{k+1}(x, m). \tag{9}$$

LEMMA 2. *If* $s(u)$ *is bounded below and summable* (A) *to* S, *then* $s(u)$ *is summable* $(C, 1)$ *to* S.

LEMMA 3. *If* $s(u)$ *is summable* (A) *to* S, *then so is* $C^\alpha(u, s)$ *for any* $\alpha > 0$.

This lemma is proved by Rajagopal ([4], Lemma 4) and independently by Szàsz ([7]. § 3 or [10], pp. 1236-7).

LEMMA 4. *If* $s(u)$ *is summable* (A), *then a necessary and sufficient condition for the convergence of* $s(u)$ *as* $u \to \infty$ *is that*

$$s(u) - (1 + c)\,C^1(u, s) \text{ converges for some real } c. \tag{10}$$

PROOF. The necessity part of the lemma is obvious. In case $c = 0$, the necessity and the sufficiency parts are together identical with the integral analogue of Tauber's second theorem ([2], Theorem 88). In case $c \neq 0$, it is enough to give a proof under the supposition that $s(u)$ is continuous in every finite interval of $u \geqslant 0$. For, when $s(u)$ does not have this property, we consider $m(u)$ defined by (7) and find from (10) that $m(u) - (1 + c)\,C^1(m, u)$ converges by (8) with $k = 1$, while the summability (A) of $s(u)$ implies that of $m(u)$ by Lemma 3. Hence, applying Lemma 4 to the continuous function $m(u)$, instead of $s(u)$, we conclude that $m(u)$ is convergent. The convergence of $s(u)$ follows now from (7) and (10).

We proceed to prove Lemma 4 for a continuous $s(u)$ and with $c \neq 0$ in (10). If we write

$$y = \int_0^x s(u)\, du$$

so that

$$\frac{dy}{dx} = s(x)$$

then (10) becomes the differential equation

$$\frac{dy}{dx} - (1 + c)\frac{y}{x} = g(x),$$

where $g(x) = l + o\,(1)$, say, as $x \to \infty$.

Solving this differential equation, we get

$$y = K x^{1+c} + x^{1+c} \int g(x)\, x^{-1-c}\, dx,$$

where K is an arbitrary constant, or

$$\begin{aligned} s(x) &= \frac{dy}{dx} = (1 + c)\frac{y}{x} + g(x) \\ &= K\,(1 + c)\, x^{c} + (1 + c)\, x^{c} \int g(x)\, x^{-1-c}\, dx + g(x) \\ &= K' x^{c} + l' + o(1),\, x \to \infty. \end{aligned}$$

If $c > 0$, then $s(x)$ clearly converges even without the assumption that $s(x)$ is summable (A). If $c > 0$, then this assumption is compatible only with $K = 0$ and hence in this case also $s(x)$ is convergent.

3. Theorems. It is convenient to prove first the following generalization of Lemma 4 and then deduce Theorem I.

THEOREM II. *If $s(u)$ is summable (A), then a necessary and sufficient condition for the summability (C, k) of $s(u)$ for a given $k > 0$ is that*

$$C^{k}(u, s) - (1 + c)\, C^{k+1}(u, s) \text{ converges for some real } c. \tag{11}$$

PROOF OF THEOREM II. Leaving out the obvious necessity part, we prove the sufficiency part as follows. It follows from (11) that

$$\frac{1}{u}\int_0^u C^{k}(x, s)\, dx - (1 + c)\,\frac{1}{u}\int_0^u C^{k}(x, s)\, dx \text{ converges.} \tag{12}$$

In virtue of (8) and (9) this means that

$$\{C^k(u, m) - C^{k+1}(u, m)\} - c\,C^{k+1}(u, m) \text{ converges,}$$

i.e.

$$\frac{1}{k+1}\{C^{k+1}(u, s) - C^{k+1}(u, m)\} - c\,C^{k+1}(u, m) \text{ converges,}$$

i.e.

$$C^{k+1}(u, s) - \{1 + (k+1)c\}\frac{1}{u}\int_0^u C^{k+1}(x, s)\,dx \text{ converges.} \quad (13)$$

Now, by Lemma 3, the summability (A) of $s(u)$ implies that of $C^{k+1}(u, s)$; and so Lemma 4 applied to $C^{k+1}(u, s)$ instead of $s(u)$, gives the convergence of $C^{k+1}(u, s)$ and (11) now yields the convergence of $C^k(u, s)$.

Proof of Theorem I. *Since $s(u)$ is summable (A)*, we have

$$C^k(u, s) - (1 + c)\,C^{k+1}(u, s) \text{ is summable } (A) \quad (14)$$

by Lemma 3. Also, the function in (14) is bounded below, by (6), and so is summable $(C, 1)$ in virtue of Lemma 2. That is, the condition (12) holds, and we are led to the convergence of $C^{k+1}(u, s)$ as in the proof of Theorem II.

4. Analogous theorems for sequences summable (A). If s is the sequence $\{s_n\} \equiv (s_0, s_1, s_2, \ldots)$ and $C_n^\alpha(s)$ is its Cesàro mean of order $\alpha > -1$, defined as usual ([2], pp. 96-7), then Theorems I and II have analogues in which $C_n^\alpha(s)$ takes the place of $C^\alpha(x, s)$. The restriction $\alpha > -1$ in the analogues, instead of the restriction $\alpha \geqslant 0$ in the original theorems given here, requires that we use, in the proofs of the analogues, the notion of summabilty (H, α) besides that of summability (c, α), the fact that $H^p C^q = C^q H^p$ in Hardy's notation ([2], p. 105) as well as the fact that summability (C, α) or equivalently summability (H, α) implies summability (A) ([2], §§ 5.2, 5.8, 5.9 and 5.12). This point is illustrated below.

Theorem I′. *If the real sequence s is summable (A) to S and if, for a $k > -1$ and some real c,*

$$C_n^k(s) - (1 + c)\ C_n^{k+1}(s) = O_L(1),\ n \to \infty, \tag{6'}$$

then s is summable $(C, k + 1)$ to S; i.e. $C_n^{k+1}(s) \to S$.

PROOF. Using the sequence-analogue of Lemma 1, we see that (6′) gives, exactly as (12) gives (13),

$$C_n^{k+1}(s) - \{1 + (k + 1)c\} \frac{C_0^{k+1}(s) + C_1^{k+1}(s) + \ldots + C_n^{k+1}(s)}{n + 1} = O_L(1).$$

Appealing now to the sequence-analogues of Lemmas 3 and 2, we get, as in the proof of Theorem I, the result that $C_n^{k+1}(s)$ is summable $(C, 1)$ or $(H, 1)$; i.e. $C_n^k(s)$ is summable $(H, 2)$ and hence summable (A). After this we argue with $C_n^k(s)$ as we have already done with $C_n^{k+1}(s)$ and conclude that $C_n^k(s)$ is summable $(C, 1)$ or that s is summable $(C, k + 1)$.

The case $k = 0$, $c = 0$ of Theorem I is due to Szàsz ([8], Satz II, [9], Theorem II; [10], pp. 560 and 626).

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ON A COMPARISON BETWEEN THE CESARO AND BOREL METHODS OF SUMMABILITY

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1. Introduction. It is well known that the Abel method of summability is strictly stronger than the Cesàro method C^r for every r, and that the Borel's (exponential) method is likewise strictly stronger than the Euler-Knopp method E_α for every α $(0 < \alpha \leqslant 1)$†; in symbols,

$$\mathfrak{A} \supset \mathfrak{C}^r, \tag{1}$$

$$\mathfrak{B} \supset \mathfrak{E}_\alpha, \tag{2}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}^r$, $\mathfrak{E}_\alpha$ denote, respectively, the sets of sequences summable by (i.e. the 'summability fields' of) the Abel, Borel, C^r- and E_α-methods respectively, and the symbol $\supset$ stands for "contains as a proper subset".

It is known also that

$$\mathfrak{A} \text{ is not a subset of } \mathfrak{E}_\alpha \text{ for any } \alpha\ (0 < \alpha < 1), \tag{3}$$

since, for instance, the series Σz^n is E_α-summable at certain points z outside the circle $|z| \leqslant 1$ while the Abel method is not even applicable to the series for points z outside the circle ([1], [2]).

On the other hand we have also ([1], § 9.8 ; [2]) :

$$\mathfrak{B} \text{ is not a subset of } \mathfrak{C}^r \text{ for any } r > 0. \tag{4}$$

The relations (3) and (4) show that, in particular, the C^r- and E_α-methods are not comparable ([1]).

However, if we confine ourselves to bounded sequences only, the situation is quite different. For bounded sequences, the Abel method is equivalent to the C^r-method for every $r > 0$, and the Borel method

† For definition and properties of the various methods of summability, see [1]. The Euler-Knopp method E_α of this note is the same as the Euler method $(E, \alpha/(1-\alpha))$ of [1] and is regular for $0 < \alpha \leqslant 1$. Throughout this note we assume that $0 < \alpha \leqslant 1$.

is equivalent to the E_α-method for every α such that $0 < \alpha < 1$ ([1]; [5]). Denoting the set of bounded sequences by $\mathfrak{S}_B$, we may express this in symbols as :—

$$\mathfrak{A} \cap \mathfrak{S}_B = \mathfrak{C}^r \cap \mathfrak{S}_B\,(r > 0), \tag{5}$$

$$\mathfrak{B} \cap \mathfrak{S}_B = \mathfrak{E}_\alpha \cap \mathfrak{S}_B\,(0 < \alpha < 1). \tag{6}$$

Also, in contrast with (3), we now have, by a result of Ramanujan [8] that

$$\mathfrak{A} \cap \mathfrak{S}_B = \mathfrak{C}^r \cap \mathfrak{S}_B \supseteq \mathfrak{E}_\alpha \cap \mathfrak{S}_B = \mathfrak{B} \cap \mathfrak{S}_B\,(r > 0,\, 0 < \alpha < 1). \tag{7}$$

The inclusion relation in (7) cannot be reversed, for examples are known of C^r-summable bounded sequences which are not Borel-summable ([1], [2], [4]). In view of (6), we have

$$\mathfrak{C}^r \cap \mathfrak{S}_B \supset \mathfrak{B} \cap \mathfrak{S}_B. \tag{8}$$

It is the object of this note to prove, by the use of functional analysis, that there is a large class of sequences which show that not only (4), but even the following sharper form of it, viz.

$$\mathfrak{B} \text{ does not contain } \bigcap_{r>0} \mathfrak{C}^r. \tag{9}$$

is true. The relation (9) will again be proved separately for the classes of bounded and unbounded sequences respectively; i.e. we shall prove that there exist sequences $\mathfrak{x}$, $\mathfrak{y}$ such that

$$\mathfrak{x} \in \bigcap_{r>0} \mathfrak{C}^r,\ \mathfrak{x} \in \mathfrak{S}_B,\ \mathfrak{x} \notin \mathfrak{B}, \tag{10}$$

$$\mathfrak{y} \in \bigcap_{r>0} \mathfrak{C}^r,\ \mathfrak{y} \notin \mathfrak{S}_B.\ \mathfrak{y} \notin \mathfrak{B}. \tag{11}$$

It may be noted that (10) is equivalent to (8), in virtue of (5), but (11) is a considerable improvement on (4).

2. Some lemmas and preliminary notions. To prove the main result of this note, given in § 3, we need the following lemmas and notions, almost all of which are well known.

LEMMA 1. ([1]). $\mathfrak{C}^r \cap \mathfrak{S}_B = \mathfrak{C}^s \cap \mathfrak{S}_B$ *for all* $r, s > 0$.

* The symbol $\supset$ denotes inclusion as a proper subset, as already pointed out.

Lemma 2. ([6]). *Let* $\Sigma\, a_n$ *be a series summable by the method* C^1. *Then the condition*

$$G: \quad a_n = 0 \ (n \neq n_k, k = 0, 1, 2, \dots)$$

will be a tauberian condition if and only if the gap G *satisfies the condition*

$$G_C: \quad n_{k+1} - n_k > \theta\, n_k$$

for some $\theta > 0$ *and for all* $k > k_0$ *where* k_0 *is some arbitrary integer.*

Remark. The meaning of the necessity part of Lemma 2 is as follows. If $\mathscr{G}_C$ denotes the set of all gaps satisfying the condition G_C and if G is any gap whatever such that $G \notin \mathscr{G}_C$, then there exists a divergent series $\Sigma\, a_n$ which has gaps given by G and which is C^1-summable.

Lemma 3. ([5]). $\mathfrak{B} \cap \mathfrak{S}_B = \mathfrak{E}_\alpha \cap \mathfrak{S}_B \quad (0 < \alpha < 1)$.

Lemma 4. ([6]). *The condition*

$$G: a_n = 0 \ (n \neq n_k, k = 0, 1, 2, \dots)$$

will be a tauberian condition for every E_α*-summable series* $\Sigma\, a_n$ *for any* α *such that* $0 < \alpha < 1$), *if the gaps* G *satisfy the condition*

$$\mathscr{G}_E: \quad n_{k+1} - n_k > \lambda \sqrt{n_k}$$

for some $\lambda > 0$ *and all* $k > k_1$.

We denote by $\mathscr{G}_E$ the set of gaps G satisfying the condition G_E.

Before we proceed to the remaining lemmas, we need the notion of *perfect* and *gap-perfect* methods of summability, introduced into the theory by Zeller and Meyer-König.

Let $H \equiv (h_{nk})$, n, $k = 0, 1, 2, \dots$, be the matrix of a summability method, and $\mathfrak{H}$ the set of sequences summable by H with $\mathfrak{H} \supseteq \mathfrak{S}_C$, the set of convergent sequences. Then H will have a certain natural topology as an FK-space ; the details may be found in Zeller's paper [9], but we do not need them in their full generality for our purpose now ; for, if H is a normal matrix (i.e. $h_{nk} = 0$ for $k > n$; $h_{nn} \neq 0$), then $\mathfrak{H}$ is a Banach space, with the norm given by

$$\|\mathfrak{x}\| = \sup_n \left| \sum_{k=0}^{n} h_{nk} x_k \right|, \quad \mathfrak{x} = \{x_k\} \in \mathfrak{H}.$$

Now, H is said to be *perfect* ([10], [9]) if $\mathfrak{S}_C$ is dense in $\mathfrak{H}$ (in the topology of $\mathfrak{H}$). For normal conservative methods H, perfectness reduces to the condition that $\Sigma |\alpha_n| < \infty$, $\Sigma \alpha_n h_{nk} = 0$ for all k imply $\alpha_n = 0$, i.e. that H be of "*type M*". It is well known that the methods C^r $(r \geqslant 0)$ and E_α $(0 < \alpha \leqslant 1)$ are perfect.

Now let G be a gap defined by the sequence of integers $\{n_k\}$, $(k = 0, 1, 2, \ldots)$. Let us form the matrix $A_G \equiv (A_G)_{ik}$ as follows:

$$(A_G)_{ik} = a_{i,n_k} + a_{i,n_k+1} + \ldots + a_{i,n_{k+1}-1}. \tag{12}$$

The matrix A, or the method A, is said to be *gap-perfect* if A_G is perfect for *every* gap G ([6]).

The lemma which follows next, is an immediate consequence of the definition of the matrix A_G.

Lemma 5. *Let $s_n = \sum_{i=0}^{n} a_i$ and let the series Σa_n have gaps G. Then the matrix method A sums $\{s_n\}$ if and only if A_G sums $\{s_{n_k}\}$.*

Lemma 6. ([6]). *The methods $C^r(r \geqslant 0)$, $E_\alpha(0 < \alpha \leqslant 1)$ are gap-perfect.*

Lemma 7. ([9]). *If a conservative (i.e. convergence-preserving) matrix method sums a divergent sequence, then it sums also an unbounded sequence.*

Lemma 8. ([7], and in more general form, [6]). *If a perfect method sums a divergent sequence, then it sums also a bounded divergent sequence.*

Lemma 9. ([6]). *Let Σb_n be a "regular" Borel-summable series, i.e. Σb_n is Borel-summable and the series $\Sigma b_n z^n$ has a radius of convergence $\rho > 0$. Then $G \in \mathscr{G}_E$ is a gap-Tauberian condition for Σb_n.*

3. Theorems. We first prove a result (Theorem 1) which is in a certain interesting relation to Lemmas 7 and 8, and which is useful in proving the main result of this note, Theorem 2. The

result in Lemma 7 had been proved, even before [9], in several weaker forms, one of which was under the additional hypothesis of the perfectness of the matrix method (e.g. [3]). Theorem 1 is a generalization of this weaker form of Lemma 7 to *FK*-spaces which are not necessarily summability fields of matrix methods.

THEOREM 1. *Let $\mathfrak{F}$ be an FK-space, with $\mathfrak{F} \supseteq \mathfrak{S}_C$ and let $\mathfrak{S}_C$ be dense in $[\mathfrak{F}]$ (in the FK-topology of $\mathfrak{F}$). Then either $\mathfrak{F} = \mathfrak{S}_C$, or $\mathfrak{F}$ is not a subset of $\mathfrak{S}_B$ and $\mathfrak{F} \cap \mathfrak{S}_B$ is not a subset of $\mathfrak{S}_C$.*

PROOF. If $\mathfrak{F} = \mathfrak{S}_C$, there is nothing to be proved. Therefore, let $\mathfrak{F} \neq \mathfrak{S}_C$, so that now $\mathfrak{S}_C \subset \mathfrak{F}$. Now, if it should be true that $\mathfrak{F} \subseteq \mathfrak{S}_B$ then

$$\mathfrak{S}_C \subset \mathfrak{F} \subseteq \mathfrak{S}_B. \tag{13}$$

We shall now show that (13) is false. For this, we note that $\mathfrak{S}_C$ is a closed subspace of the *FK*-space $[\mathfrak{S}_B]$, and convergence in $[\mathfrak{S}_C]$ implies convergence in $[\mathfrak{F}]$, and convergence in $[\mathfrak{F}]$ implies convergence in $[\mathfrak{S}_B]$ (assuming (13) to be true), by a theorem of Zeller on comparable *FK*-spaces ([9], Satz 4.5). Hence if (13) be true and there be $\mathfrak{x}$ such that

$$\mathfrak{x} \in \mathfrak{F} \subseteq \mathfrak{S}_B, \ \mathfrak{x} \notin \mathfrak{S}_C,$$

then $\mathfrak{x}$ is a contact point of $\mathfrak{S}_C$ in $[\mathfrak{F}]$, but not of $\mathfrak{S}_C$ in $[\mathfrak{S}_B]$ since $\mathfrak{S}_C$ is closed in $[\mathfrak{S}_B]$; by the previous remark this cannot happen if $\mathfrak{F} \subseteq \mathfrak{S}_B$. We have thus proved that $\mathfrak{F}$ is not a subset of $\mathfrak{S}_B$.

Finally, let $\mathfrak{e} = (1, 1, 1, \ldots)$ and $\mathfrak{e}_i = (0, \ldots, 0, 1, 0, \ldots)$ where all except the ith co-ordinate are zero. Then it is well known that the points $\mathfrak{e}, \mathfrak{e}_0, \mathfrak{e}_1, \ldots$ form a fundamental set (Grundmenge) in $[\mathfrak{S}_C]$; that is, if Δ denotes the set of all finite linear combinations of $\mathfrak{e}, \mathfrak{e}_0, \mathfrak{e}_1, \ldots$, then Δ is dense in $[\mathfrak{S}_C]$, in the $\mathfrak{S}_C$-topology; since $\Delta \subset \mathfrak{S}_C \subseteq \mathfrak{F}$ it follows again from the theorem on comparable *FK*-spaces that Δ is dense in $\mathfrak{S}_C$ in the $\mathfrak{F}$-topology. Since $\mathfrak{S}_C$ is dense in $[\mathfrak{F}]$, it follows now Δ is also dense in $[\mathfrak{F}]$. A result of Meyer-König and Zeller ([6], Satz 3.1) now gives immediately the result that $\mathfrak{F} \cap \mathfrak{S}_B \subset \mathfrak{S}_C$. This completes the proof of the theorem.

THEOREM 2. *Corresponding to every gap G such that $G \in \mathscr{G}_E$, $G \notin \mathscr{G}_C$ [that is, G being given by the sequence $\{n_k\}$, $n_{k+1} - n_k > \lambda\sqrt{n_k}$ for some $\lambda > 0$ and all $k > k_0$ but there is no $\theta > 0$ such that $n_{k+1} - n_k > \theta\, n_k$ for all $k > k_1$ (where k_0, k_1, are arbitrary integers)], there is a series $\Sigma\, a_n$ with bounded partial sums and a series $\Sigma\, b_n$ with unbounded partial sums such that each series has gaps G, and is C^r-summable for every $r > 0$ but is not Borel-summable.*

PROOF. Let C^r_G denote the matrix formed from the C^r-matrix by means of the equations (12), where G is any arbitrary fixed gap of the type mentioned in the hypothesis. Let $\mathfrak{C}^r_G$ denote the summability field of C^r_G. By the necessity part of Lemma 2, there exists a series $\Sigma\, u_n$, with gaps G and partial sums x_n such that

$$\{x_n\} = \mathfrak{x} \in \mathfrak{C}^1,\ \mathfrak{x} \notin \mathfrak{S}_C, \text{ i.e. } \mathfrak{S}^1_G \supset \mathfrak{S}_C.$$

It follows from Lemmas 6, 8 that there exists a series $\Sigma\, a_n$ with partial sums y_n such that

$$\{y_n\} = y \notin \mathfrak{S}_C,\ \mathfrak{y} \in \mathfrak{C}^1_G \cap \mathfrak{S}_B.$$

But

$$\mathfrak{C}^1_G \cap \mathfrak{S}_B = \mathfrak{C}^r_G \cap \mathfrak{S}_B, \qquad \text{by Lemmas 1, 5,}$$

$$= \bigcap_{r>0}\{\mathfrak{C}^r_G \cap \mathfrak{S}_B\} = \{\bigcap_{r>0}\mathfrak{C}^r_G\} \cap \mathfrak{S}_B,$$

and hence we see that $\mathfrak{y} \in \bigcap_{r>0} \mathfrak{C}^r_G \cap \mathfrak{S}_B$. Now, since $\Sigma\, a_n$ has gaps $G \in \mathscr{G}_E$ and $\mathfrak{y} \notin \mathfrak{S}_C$, it follows from Lemma 4 that it is not E_α-summable for any $\alpha\,(0 < \alpha \leqslant 1)$, so that, Lemma 3 now gives $\mathfrak{y} \notin \mathfrak{B}$. This proves the first part of the theorem.

[It follows from Lemma 7 applied to C^1_G that there exist C^1-summable series with unbounded partial sums and with gaps G, which are not E_α-summable for any α. The theorem however asserts much more than this].

To prove the second part, we write

$$\mathfrak{F} = \bigcap_{n>1} \mathfrak{C}^{1/n}_G \tag{14}$$

where G is any gap of the type being considered. Since

$$C^r_G \subset C^s_G \text{ for } r < s, \tag{15}$$

we see that in fact $\mathfrak{F} = \bigcap_{r>0} C^r_G$.

By Lemma 6 on the gap-perfectness of C^r, we have:

$\mathfrak{S}_G$ is dense in each of the FK-spaces $[\mathfrak{C}^{1/n}_G]$ in the respective topology of each $[\mathfrak{C}^{1/n}_G]$.

Using (15) and the theorem on comparable FK-spaces ([9], Satz 4), we see that $\mathfrak{S}_G$ is dense in $[\mathfrak{F}]$, in the FK-topology of $\mathfrak{F}$, which is the lattice-product of the topologies of the spaces $[\mathfrak{C}^{1/n}_G]$. We have already proved, in the first part, that there exists a $\mathfrak{y}$ such that

$$\mathfrak{y} \notin \mathfrak{S}_G,\ \mathfrak{y} \in \bigcap_{r>0} C^r_G \cap \mathfrak{S}_B = \mathfrak{F} \cap \mathfrak{S}_B.$$

Applying now Theorem 1 to the FK-space $[\mathfrak{F}]$, we get that

$$\mathfrak{F} \text{ is not contained in } \mathfrak{S}_B.$$

That is, there exists an unbounded sequence $\mathfrak{u} \in \mathfrak{F}$, so that then there exists a series with unbounded partial sums and which is summable by every one of the methods C^r_G $(r > 0)$. This means, in virtue of Lemma 5, that there exists a series $\Sigma\, b_n$, say, whose partial sums are unbounded and which has gaps G and is C^r-summable for every $r > 0$. We shall show that $\Sigma\, b_n$ is not Borel-summable.

To see this, since $\Sigma\, b_n$ is C^r-summable, then by a classical result we must have $b_n = o(n^r)$. The series $\Sigma\, b_n z^n$ therefore has a positive radius of convergence. The desired conclusion now follows immediately from Lemma 9.

4. Some further remarks. (1). Lemma 2 mentions explicitly only the C^1-method, but it is easy to see that G_C is a necessary and sufficient gap-tauberian condition (in the sense of the Remark after Lemma 2) for the C^r-method $(r > 0)$ and the Abel method. Sufficiency is obvious since G_c is tauberian even for Abel's method. If $G \notin \mathscr{G}_c$ then there exists $\mathfrak{y}$ such that $\mathfrak{y} \in \mathfrak{C}^r_G \cap \mathfrak{S}_B$, for every $r > 0$; this proves the necessity part.

(2). The following example given by Knopp [4] is a particular case of the method discussed in this note :

$$\mathfrak{x} = \{x_n\} = \{\sum_{i=0}^{n} a_i\}, \text{ with } a_0 = 0,\ a_{k^2} = (-1)^{k+1},\ a_n = 0\ (n \neq k^2).$$

This corresponds to gap-series with gaps G given by $\{n_k\}$, where $n_k = k^2$, when $G \in \mathscr{G}_B$, $G \notin \mathscr{G}_C$ and is of the type considered in Theorem 2.

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A NOTE ON THE EQUIDISTRIBUTION OF SUMS OF INDEPENDENT RANDOM VARIABLES

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1. Introduction. Let $X_1, X_2, \ldots$ be a sequence of independent real-valued random variables with a common distribution function $F(x) = Pr(X_n \leqslant x)$. Let $S_n = X_1 + \ldots + X_n$. Let $\phi(t) = \int e^{itx} dF(x)$. With respect to the roots of the equation $\phi(t) = 1$, there are three cases to consider according as $\phi(t) = 1$ only when $t = 0$, $\phi(t) = 1$ for all t and $\phi(t) = 1$, for some $t \neq 0$. In the last case (called the lattice case) it can be shown that there is a suitable $\beta > 0$ such that $\phi(t) = 1$ if and only if $t = 2k\pi/\beta$, $(k = 0, \pm 1, \ldots)$. Corresponding to these three cases we define the mean value $M(h)$ (when it exists) of a function $h(x)$ as

$$M(h) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(x)\, dx,$$

$$M(h) = h(0)$$

and

$$M(h) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{j=-N}^{N} h(j\beta)$$

respectively.

It can be shown that when $h(x) = e^{itx}$, in all cases $M(h) = 1$, if $\phi(t) = 1$ and 0 otherwise.*

Robbins [1] proved the following

THEOREM 1.1. *The class H of functions $h(x)$ such that with probability one*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(S_j) = M(h) \tag{1.1}$$

* I am grateful to the referee for suggesting the simplified characterization of $M(\exp itx)$ in all cases.

contains all uniformly almost periodic functions. Also if $h(x)$ has period p and is Riemann integrable in the interval $0 \leqslant x \leqslant p$ then with probability one

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(S_j) = \frac{1}{p} \int_{0}^{p} h(x)\, dx, \tag{1.2}$$

provided the first case holds or the last case with β not a rational multiple of p.

2. We now examine the validity of (1.1) when we replace the S_j's by X_j's themselves.

For $t = 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} = M(e^{itx}) = 1. \tag{2.1}$$

If $t \neq 0$, but $\phi(t) = 1$, we obtain the lattice case with $t = 2k\pi/\beta$. Hence $M(e^{itx}) = 1$.

Since each X_n is an integral multiple of β, $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} = 1$.

Hence in this case also

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} = M(e^{itx}) = 1. \tag{2.2}$$

We have to consider the only remaining case when $\phi(t) \neq 1$, so that $|\phi(t)| < 1$. Put

$$Y_n = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j}. \tag{2.3}$$

Then

$$E(Y_n) = \phi(t). \tag{2.4}$$

Consider

$$Z_n = Y_n - \phi(t) = \frac{1}{n} \sum_{j=1}^{n} [\, e^{itX_j} - \phi(t)]. \tag{2.5}$$

Then

$$|Z_n|^2 = Z_n \cdot \bar{Z}_n = \frac{1}{n^2}\sum_{j=1}^{n} [e^{itX_j} - \phi(t)][e^{-itX_j} - \overline{\phi(t)}] +$$

$$+ \frac{1}{n^2}\sum_{\substack{j,k=1 \\ (j\neq k)}}^{n} [e^{itX_k} - \phi(t)][e^{-itX_j} - \overline{\phi(t)}] \tag{2.6}$$

Hence

$$E[|Z_n|^2] = \frac{1}{n}[1 - \phi(t)\ \overline{\phi(t)}] \tag{2.7}$$

$$\leqslant \frac{\lambda}{n} \text{ for some finite constant } \lambda, \tag{2.8}$$

since $|\phi(t)| < 1$.

Hence, for any $\epsilon > 0$

$$Pr\ [|Z_n| > \epsilon] \leqslant \frac{E[|Z_n|^2]}{\epsilon^2} \leqslant \frac{\lambda}{n\,\epsilon^2}. \tag{2.9}$$

Therefore

$$\sum Pr\ [|Z_{n^2}| > \epsilon] < \infty. \tag{2.10}$$

By applying Borel Cantelli lemma

$$Pr\ [\lim_{n\to\infty} Z_{n^2} = 0] = 1. \tag{2.11}$$

For any positive integer n, we can choose an $m(n)$ such that

$$m^2 \leqslant n < (m+1)^2. \tag{2.12}$$

So

$$Z_n = \frac{1}{n}\sum_{j=1}^{n} [e^{itX_j} - \phi(t)] = \frac{m^2}{n} Z_{m^2} + \frac{1}{n}\sum_{j=m^2+1}^{n} [e^{itX_j} - \phi(t)]. \tag{2.13}$$

Hence

$$\left| Z_n - \frac{m^2}{n} Z_{m^2} \right| \leqslant \frac{2(n-m^2)}{n} \leqslant 2\,\frac{(2m+1)}{m^2}. \tag{2.14}$$

So, as $n \to \infty$

$$\left| Z_n - \frac{m^2}{n} Z_{m^2} \right| \to 0. \tag{2.15}$$

Since $m^2/n \to 1$ as $n \to \infty$,

$$Pr\,[\lim_{n\to\infty} Z_n = 0] = 1. \tag{2.16}$$

So

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{itX_j} = \phi(t) \text{ with probability one.} \tag{2.17}$$

Since Y_n does not tend to zero as $n \to \infty$ we find that (1.1) is not valid when S_j's are replaced by X_j's.

3. We now examine a case when S_j's are taken in a different way with respect to the sequence $\{X_j\}$.

Let $p_1 < p_2 \ldots < p_n < \ldots$ be a sequence of integers which tends to infinity. Let

$$S_1 = X_1 + X_2 + \ldots + X_{p_1},$$
$$S_2 = X_{p_1+1} + \ldots + X_{p_1+p_2},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_n = X_{p_1+\ldots+p_{n-1}+1} + \ldots + X_{p_1+\quad+p_n}. \tag{3.1}$$

THEOREM 3.1. *For any $F(x)$ and any real number t (as defined in § 1), the relation*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{itS_j} = M(e^{itx}) \tag{3.2}$$

holds with probability one.

PROOF. For $t = 0$, both sides of (3.2) are equal to 1. If $t \neq 0$, but $\phi(t) = 1$, we obtain the lattice case with $t = 2k\pi/\beta$. Hence the right side of (3.2) is equal to one. Since each X_n and hence each S_n is an integral multiple of β, the left side of (3.2) is also equal to 1. We have to consider only the remaining case $\phi(t) \neq 1$ so that $|\phi(t)| < 1$. In this case we must show that the left side of (3.2) has value 0. Let

$$Y_n = \frac{1}{n} \sum_{j=1}^{n} e^{itS_j} \tag{3.3}$$

$$|Y_n|^2 = Y_n . \bar{Y}_n \tag{3.4}$$

$$= \frac{1}{n^2} \sum_{j,k=1}^{n} e^{it(S_k - S_j)} \tag{3.5}$$

$$= \frac{1}{n} + \frac{1}{n^2} \sum_{\substack{j,k=1 \\ (j \neq k)}}^{n} e^{it(S_k - S_j)} \tag{3.6}$$

$$E[|Y_n|^2] = \frac{1}{n} + \frac{1}{n^2} \sum_{\substack{j,k=1 \\ (j \neq k)}}^{n} [\phi(t)]^{p_k} [\overline{\phi(t)}]^{p_j} \tag{3.7}$$

$$= \frac{1}{n} + \frac{\lambda}{n^2}, \tag{3.8}$$

where $\lambda < \dfrac{1}{[1 - |\phi(t)|]^2}$ since $|\phi(t)| < 1$ and p_i is an increasing sequence. Hence we can write

$$E[|Y_n|^2] < \frac{A}{n}, \text{ where } A \text{ is some finite constant.} \tag{3.9}$$

Hence by using arguments similar to those in §2 we get

$$Pr[\lim_{n \to \infty} Y_n = 0] = 1.$$

4. By using the same reasoning as that employed by Robbins we get a theorem which is the same as Theorem 1.1 for the S_j's defined in § 3.

5. Application to Stochastic process with stationary increments.

If $X(t)$ denotes a stochastic process with stationary increments, define

$$\theta(t) = X\left(\frac{t(t+1)}{2}\right) - X\left(\frac{(t-1)t}{2}\right).$$

Now $\theta(1)$, $\theta(2)$,... correspond to S_1, S_2,... in § 3 with $p_1 = 1$, $p_2 = 2, \ldots, p_j = j, \ldots$. Then the following theorem is true.

THEOREM 5.1. *If $g(x)$ is a uniformly almost periodic function then* $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} g\{\theta(i)\} = M(g)$ *with probability one. If $g(x)$ has period p and is Riemann integrable in $0 \leqslant x \leqslant p$, then with probability one*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g\{\theta(i)\} = \frac{1}{p} \int_0^p g(x)\, dx.$$

6. It is particularly important to note that the theorems given in §§ 3 and 4 will be true only if the S_j's are such that $p_1 < p_2 < \ldots . p_n < \ldots .$ If we take the sequence $\{p_i\}$ to be bounded, the theorems may not hold. For instance, consider $p_1 = p_2 = \ldots, = p_n = \ldots, = 1$. Then the problem reduces to the one discussed in § 2 and it is evident that the theorem does not hold.

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ORTHOGONALITY IN NORMED LINEAR SPACES

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THE notion of angle between two elements of finite dimensional euclidean spaces has been extended in a natural way to inner product spaces, viz. the angle between x, y is ϕ if $\cos\phi = |(x, y)|/\|x\|\,\|y\|$. No attempt seems to have been made to define angle in non-inner product normed linear spaces. However, orthogonality has been defined by Roberts [4] and James [1, 2]. In this paper the angle between the elements has been defined and this definition has the advantage of being closely connected with functionals and hyperplanes.

We shall use the following notation. B represents a real normed linear space; $x, y, z, \ldots$ are non-null elements of B and $a, b, c, \ldots$ are real scalars. $\langle x, y\rangle = \theta$ means that x is inclined to y at an angle θ.

DEFINITION 1. *Let S be a closed linear subspace of B, $\langle x, S\rangle = \theta$ if and only if*

$$\operatorname*{g.l.b.}_{h\in S} \frac{\|x+h\|}{\|x\|} = \lambda, \text{ where } \lambda = \sin\theta.$$

It is obvious that $0 \leqslant \lambda \leqslant 1$. If S is a closed linear subspace of B, generated by an element y, then it follows that $\langle x, y\rangle = \theta$ if and only if $\operatorname*{g.l.b.}_{k} \frac{\|x+ky\|}{\|x\|} = \lambda$. If $\lambda = 1$ this definition reduces to orthogonality of James [2]. That this is a natural extension of the notion of angle in inner product spaces is seen from the following theorem.

THEOREM 1. *In inner product spaces*

$$\langle x, y\rangle = \cos^{-1}\frac{|(x, y)|}{\|x\|\,\|y\|} = \langle y, x\rangle.$$

Proof. Let the function $F(k)$ be defined by

$$F(k) = \frac{\|x+ky\|^2}{\|x\|^2} = 1 + k^2 \frac{\|y\|^2}{\|x\|^2} + 2kt \frac{\|y\|}{\|x\|},$$

where $t = \dfrac{(x, y)}{\|x\| \, \|y\|}$.

It is easily seen that $F(k)$ attains the minimum value $\sin^2\phi$ when $k = -t$. Hence

$$\langle x, y \rangle = \underset{k}{\text{g. l. b}} \, |\sqrt{F(k)}| = \underset{k}{\text{g. l. b.}} \frac{\|x+ky\|}{\|x\|}$$

$$= \phi = \cos^{-1} \frac{|(x, y)|}{\|x\| \cdot \|y\|}.$$

Similarly it can be shown that $\langle y, x \rangle = \phi$.

In general normed linear spaces the angle defined in (1) is homogeneous, i.e. $\langle x, y\rangle = \langle ax, \; by \rangle$ but not necessarily symmetric, $\langle x, y \rangle \neq \langle y, x \rangle$. It may be mentioned in this connection that James [3] has proved that orthogonality is symmetric in a normed linear space if and only if an inner product can be induced by the given norm the well-known condition for which is

$$\|x+y\|^2 + \|x-y\|^2 = 2\,[\|x\|^2 + \|y\|^2].$$

Similarly, it can be shown, more generally, that for a fixed θ, $\langle x, y \rangle = \theta$ implies $\langle y, x\rangle = \theta$ if and only if the space has the above property.

Remark. For a given element x, the elements y, $\langle x, y\rangle = \theta$ form a closed linear subset $C \subset B$. For, let y_n $(n = 1, 2, 3, \ldots)$ be a sequence of elements such that $\langle x, y_n \rangle = \theta$.

Then

$$\underset{k}{\text{g. l. b.}} \frac{\|x + k\,y_n\|}{\|x\|} = \lambda, \text{ where } \lim_{n\to\infty} \|y_n - y_0\| = 0.$$

Hence $y_0 \in C$. Also if $y \in C$, $a\,y \in C$. Hence C is a cone.

LEMMA 1. *If S is a closed linear subspace of B, and $x \notin S$ and ϕ is any linear functional such that $\phi(h) = 0$, $h \in S$ and $\|\phi\| = 1$, then $|\phi(x)| \leqslant \rho(x, S)$, where $\rho(x, S)$ denotes the distance between x and S.*

PROOF. The set of functionals with the above property is non-null for the Hahn-Banach theorem guarantees the existence of a linear functional f, such that $f(x) = \rho(x, S)$

$$\|f\| = 1, \text{ and } f(h) = 0,\ h \in S.$$

Now if ϕ were to be any other functional with the property assumed in the lemma, then

$$|\phi(x+h)| = |\phi(x)| \leqslant \|x+h\| \cdot \|\phi\|, \text{ for } h \in S,$$

since $\|\phi\| = 1$, $\phi(x) \leqslant \underset{h \in S}{\text{g. l. b.}} \|x+h\| = \rho(x, S)$.

2. The following theorem is fundamental.

THEOREM 2. *Let f be a given linear functional, $\|f\| \neq 0$, and H be the hyperplane defined by $f(h) = 0$. Then $\langle x, H \rangle = \theta$ if and only if $|f(x)| = \lambda \|x\| \cdot \|f\|$, $\lambda = \sin\theta$.*

PROOF: Suppose $\langle x, H \rangle = \theta$, then

$$\rho(x, H) = \underset{h \in H}{\text{g. l. b}} \|x+h\| = \lambda \|x\|. \tag{1}$$

Let μ be defined by $|f(x)| = \mu \|x\| \cdot \|f\|$. Since $h \in H$

$$|f(ax+h)| = |a| \mu \|x\| \cdot \|f\| \leqslant |a| (\mu/\lambda) \|x + h/a\| \cdot \|f\|$$
$$= (\mu/\lambda) \|ax+h\| \cdot \|f\|. \tag{2}$$

Let $y \in B$; since H is a hyperplane, $y = ax + h$, $h \in H$ and for some real scalar a. Hence, for every $y \in B$

$$|f(y)| \leqslant (\mu/\lambda) \|y\| \|f\|, \text{ from (2)}.$$

Hence

$$\mu/\lambda \geqslant 1. \tag{3}$$

Again

$$\left|\frac{f(x)}{\|f\|}\right| \leqslant \rho(x, H) = \lambda \|x\|, \text{ by Lemma 1.}$$

Hence, putting $|f(x)| = \mu \, \|x\| . \|f\|$, we obtain

$$\mu/\lambda \leqslant 1. \tag{4}$$

From (3) and (4) it follows that $\mu = \lambda$.

Conversely if $|f(x)| = \lambda \, \|x\| . \|f\|$ then since such a λ is unique, $\langle x, H \rangle = \theta$, where H is the hyperplane defined by $f(h) = 0$ and $\lambda = \sin \theta$.

If in the above definition we replace the hyperplane by a closed linear subspace, then it is not possible to obtain a similar result. However we can prove

THEOREM 3. *If S is a closed linear subspace of B, and $x \perp S$ then there exist elements y, such that $\langle y, S \rangle = \theta$.*

PROOF. We can assume that $\|x\| = 1$, without any loss of generality. Let $y = \lambda x + \mu h$, where $\lambda = \sin \theta$ and $\|\lambda x + \mu h\| = 1$. That there is such an element y follows from Lemma 2 given below. The element y depends on the choice of $h \in S$. For any $h' \in S$, we have

$$\|y + h'\| = \|\lambda x + \mu h + h'\| = |\lambda| . \left\| x + \frac{\mu h + h'}{\lambda} \right\|$$

since

$$\frac{\mu h + h'}{\lambda} \in S, \text{ and } x \perp S, \ \lambda \left\| x + \frac{\mu h + h'}{\lambda} \right\| \geqslant \lambda \, \|x\|$$

and the equality arises when $(\mu h + h')/\lambda$ is the null element. Hence $\underset{h' \in S}{\text{g.l.b.}} \|y + h'\| = \lambda \|x\| = \lambda \|y\|$, and it follows that $\langle y, S \rangle = \theta$. The set of elements y, such that $\langle y, S \rangle = \theta$ form a cone.

James [2] has given a necessary and sufficient condition that there exists an element orthogonal to each closed linear subspace. By Theorem 3 the following result is easily obtained.

THEOREM 4. *The necessary and sufficient condition that there exists a cone C, with vertex at the origin such that every element of the cone is inclined at a given angle to a given closed linear subspace of B, is that for each linear functional f, defined in B, there exists an element x, such that $|f(x)| = \|f\| \cdot \|x\|$.*

For any linear functional f defined in an abstract euclidean space E, there exists a unique element on the unit sphere such that $|f(x)| = \|x\| \cdot \|f\|$. Hence, for each closed linear subspace there is a unique cone with vertex at the origin, the elements of the cone are inclined at a given angle θ with S.

It is easily seen from the results of James ([3], Theorems 7.2, 7.3 and 7.4) that if (i) the unit sphere of B is weakly compact, or (ii) the Banach space B is regular or if the norm of B is uniformly Frechet differentiable or (iii) the Banach space B is uniformly convex, then there exist cones of the type referred to above, associated with each closed linear subspace of B.

3. Given x, the question arises whether there necessarily exists an element y inclined at a given angle θ, i.e. $\langle x, y \rangle = \theta$. That this is so is shown by

LEMMA 2. *Given that* $x \perp h$, *there exist elements of the form* $\lambda x + \mu h$ *inclined at an angle* $\sin^{-1} \lambda$ *with* h.

PROOF. There is no loss of generality in assuming that $\|x\| = \|h\| = 1$. We shall establish first that given $0 \leqslant \lambda \leqslant 1$ we can find an element $\lambda x + \mu h$ on the unit sphere, where μ is a function of λ. Consider the elements algebraically between x and h, i.e. elements of the form $ax + bh$, $a \geqslant 0$, $b \geqslant 0$, and $a + b = 1$. Then the radius through $ax + bh$ meets the unit sphere at $\dfrac{(ax + bh)}{\|ax + bh\|}$

Since x and h are linearly independent (Lemma 3)

$$\lambda = \frac{a}{\|ax + bh\|}, \quad \mu = \frac{b}{\|ax + bh\|}.$$

Consider the function $F(a)$ defined by

$$F(a) = \frac{a}{\|ax + (1 - a)h\|}$$

then $F(0) = 0$, and $F(1) = 1$.

Since $F(a)$ is a continuous function of a, given λ, $0 \leqslant \lambda \leqslant 1$, there exists a value a_0 of a such that $F(a_0) = \lambda$.

Further $\langle \lambda x + \mu h, h \rangle = \sin^{-1} \lambda$, for since $\langle x,h \rangle = \pi/2$ there exists a linear functional f such that $f(h) = 0$, $f(x) = \|x\| \, \|f\|$.

Hence

$$|f(\lambda x + \mu h + kh)| = |\lambda f(x)| \leqslant \|\lambda x + \mu h + kh\| . \|f\|$$

for real k, and for $k = -\mu$ the equality arises. Hence,

$$\langle \lambda x + \mu h, h \rangle = \sin^{-1} \lambda.$$

Hence given θ, let $\lambda = \sin \theta$. Corresponding to this value of $F(a)$, let the value of a be a_0. Consider the element $ax + bh$, then the radius through this element intersects the unit sphere at y which has the property $\langle y, h \rangle = \theta$.

Corollary. *If orthogonality is symmetric, i.e.* $x \perp h$ *implies* $h \perp x$, *then* $\langle y, x \rangle = \sin^{-1} \mu$.

In inner product spaces $x \perp y$ if $(x, y) = 0$. It is well known that in such spaces x and y are linearly independent. It is an interesting fact that this continues to hold good under the Definition 1.

Lemma 3. *If* $x \perp y$ *then* x *and* y *are linearly independent.*

Proof. If possible let x and y be linearly dependent, i.e. $ax + by = 0$, where a and b are not both zero. Since $x \perp y$ there exists a linear functional f such that

$$f(y) = 0, f(x) = \|f\| . \|x\|;$$

$$f(ax + by) = a f(x) = a \; \|f\| . \|x\| = 0.$$

Hence $a = 0$ and $by = 0$ which is impossible unless $b = 0$ since y is non-null. Hence x and y are linearly independent.

Theorem 4. *The necessary and sufficient condition that* $\lambda x + \mu h$ *is inclined at an angle* $\sin^{-1} \lambda$ *with* h, *where* $\|\lambda x + \mu h\| = \|x\| = \|h\| = 1$ *is that* $x \perp h$.

Proof. The condition is necessary, for, if $\langle \lambda x + \mu h, h \rangle = \sin^{-1} \lambda$ then

$$\underset{k}{\text{g. l. b.}} \frac{\|\lambda x + \mu h + kh\|}{\|\lambda x + \mu h\|} = \lambda,$$

i.e. $\rho(\lambda x + \mu h, h) = \lambda \|\lambda x + \mu h\| = \lambda$, since $\|\lambda x + \mu h\| = 1$.

Hence by Hahn-Banach theorem there exists a linear functional f such that $f(h) = 0$, $\|f\| = 1$ and $f(y) = \lambda \|y\|$, where $y = \lambda x + \mu h$ and hence

$$f(\lambda x + \mu h) = f(y) = \lambda \|y\| = \lambda \|x\|.$$

Hence $f(\lambda x + \mu h) = \lambda f(x) = \lambda \|x\|$, or $f(x) = \|x\| . \|f\|$ and $f(h) = 0$. Hence $x \perp h$.

Again if $x \perp h$, then

$$\|\lambda x + \mu h + kh\| = \lambda \left\| x + \frac{\mu h + kh}{\lambda} \right\| \geqslant \lambda \|x\|$$

and equality arises when $k = -\mu$. Hence

$$\langle \lambda x + \mu h, h \rangle = \sin^{-1} \lambda.$$

REMARK. In view of the above theorem we can assert that corresponding to each property of orthogonality we can state a property of elements inclined at an angle θ as defined here. In particular we can assert the following analogues of James [2].

THEOREM 5. *Given any two elements x and y there exist elements $px + y$, such that $\langle px + y, x \rangle = \theta$.*

PROOF. It is enough to observe that given x and y there is an element $z = ax + y \perp x$. Once this element is known, then by Lemma 2 we can find an element $rz + x$, such that $\langle rz + x, x \rangle = \theta$.

But

$$rz + x = r(ax + y) + x = (ar + 1)x + ry = r \left[\frac{ar + 1}{r} x + y \right].$$

Since the angle is homogeneous $\frac{ar + 1}{r} x + y$, which is of the form $px + y$, is inclined to x at an angle θ. Also by Theorem 5, given x and y we can find as many elements of the form $px + y$, such that $\langle px + y, x \rangle = \theta$ as there are of the type $ax + y$, such that $ax + y \perp x$. In fact there is a one-one correspondence between the elements $px + y$ and $ax + y$ referred to above.

If the norm is strictly convex James [2] has shown that the element $ax + y$ for which $ax + y \perp x$ is unique. Hence in such spaces given x and y there is a unique element $px + y$, such that $\langle px + y, x \rangle = \theta$.

Theorem 6. *Let $\|x\| = \|y\| = 1$ and $ax + y$ be the element such that $ax + y \perp x$. Then $\langle y, x \rangle = \sin^{-1} \|ax + y\|$.*

Proof. First we shall show that $\|ax + y\| \leqslant 1$. Since $ax + y \perp x$ there exists a linear functional such that

$$f(ax + y) = \|f\| . \|ax + y\| \text{ and } f(x) = 0.$$

Hence $f(ax + y) = f(y) \leqslant \|y\| . \|f\|$ and hence $\|ax + y\| \leqslant \|y\| = 1$.

Consider the element $h = \dfrac{ax + y}{\|ax + y\|}$. Then $\|x\| = \|h\| = \|y\| = 1$ and $h \perp x$ and $y = h \|ax + y\| - ax$.

Hence $\langle y, x \rangle = \sin^{-1} \|ax + y\|$ which completes the proof.

In the other direction if the norm is Gateux differentiable, i.e. $\lim\limits_{h \to 0} \dfrac{\|x + hy\| - \|x\|}{h} = a \|x\|$ exists for non-null x and y, then there is a unique element $-ax + y$ such that $x \perp -ax + y$.

Further it can be shown that, if $\|x\| = \|y\| = 1$ and $x \perp y - ax$ then $\langle y, y - ax \rangle = \sin^{-1} a$.

Since $x \perp y - ax$ there exists a linear functional f such that $f(x) = \|x\| . \|f\|$ and $f(y - ax) = 0$, i.e. $f(y) = af(x)$. Hence $f(y) = a \|x\| . \|f\| \leqslant \|y\| . \|f\|$ and $a \leqslant 1$, since $\|x\| = \|y\| = 1$.

Now consider $h = \dfrac{y - ax}{\|y - ax\|}$, then $\|x\| = 1$ and $x \perp h$ and $y = h\|y - ax\| + ax$. Hence, as in Theorem 6, $\langle y, h \rangle = \sin^{-1} a$.

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CLASSIFICATION OF ALGEBRAS

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1. Introduction. This paper was motivated while working together with Prof. S. Minakshisundaram on another work entitled H^*-algebras. The system considered is an algebra A with a unit over the field C of complex numbers. It is also an inner product space, under the norm topology of which the left and right multiplications are continuous. We do not assume at this stage that A is complete.

Now two cases arise: (i) either the completed space $\bar{A}$ can be made an algebra so that the left and right multiplications are continuous and continuous extensions of those already in A, or (ii) $\bar{A}$ cannot be made such an algebra.

There are two cases about the $\|x\|$ in A, viz. (a) either there exists a constant K such that $\|xy\| \leqslant K\|x\|\,\|y\|$ for all $x, y \in A$ or (b) there is no such K.

Now I have proved in this paper that the statement that there exists a constant K, such that $\|xy\| \leqslant K\|x\|\,\|y\|$ for all $x, y \in A$ is equivalent to saying that A can be completed into an algebra $\bar{A}$ as given in case (i) above.

A more general result than this is proved, viz.: If H is an algebra and also a Hilbert space with an involution * such that $(ab, c) = (b, a^*c) = (a, cb^*)$ for all $a, b, c \in H$ and has an identity, then H is finite dimensional and is isomorphic to the direct product of a finite number of the algebra of matrices of some order 'n' led by the trace topology.

The notations used are as given in [3] and [4].

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2. DEFINITION 1. *A set H of elements is said to be a Hilbert algebra or an H-algebra if the following conditions are fulfilled.*

1. *H is a Hilbert space which is complete uuder the Hilbert norm $\|x\| = \sqrt{(x,x)}$, where $x \in H$.*

2. *H becomes an algebra over the fie!d C of complex numbers when it is led by the same operations of addition and scalar multiplication as demanded by* (i) *and another ring multiplication.*

3. *There is an involution defined in H, i.e. to every element $a \in H$ there corresponds an element $a^* \in H$ such that*

(a) $(a^*)^* = a$ *for all* $a \in H$,

(b) $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$ *for all* $a, b \in H$ *and* $\lambda, \mu \in C$,

(c) $(ab, c) = (b, a^*c) = (a, c b^*)$ *for all* $a, b, c \in H$.

4. *The operators $x \rightarrow ax$ and $x \rightarrow xa$ are bounded operators for each $a \in H$.*

5. *H has a unit which will be denoted by* 1 *with regard to multiplication.*

LEMMA 1. $(x, y) = (y^*, x^*)$ *for all* $x, y \in H$ *and* $x x^* = 0 \rightleftarrows x = 0$.

DEFINITION 2. *An element $x \in H$ will be said to be 'self-adjoint' if $x = x^*$.*

An element $e \in H$ will be said to be 'idempotent' if $0 \neq e = e^2$.

*An element $e \in H$ will be said to be a '*sa-*idempotent' (self-adjoint idempotent) if $0 \neq e = e^* = e^2$.*

Let e_1 and e_2 be two sa-*idempotents. Then we say $e_1 \geqslant e_2$ if and only if $e_1 e_2 = e_2 e_1 = e_2$.*

REMARK 1. The $\geqslant$ defined in the set of sa-idempotents is a partial ordering relation.

NOTATION. We will use l_a and r_a to denote the operators $x \rightarrow ax$ and $x \rightarrow xa$ in H (only if a is an element of H). l_a and r_a have the same bound or norm, i.e. $|||l_a||| = |||r_a|||$. We will mention here without proof the following results proved in [2].

THEOREM 1. *To every projection operator P in H whose range is a closed right ideal in H, there corresponds a unique* sa-*idempotent e such that $P = l_e$ and conversely.*

THEOREM 2. *Every self-adjoint element $x \in H$ can be expressed as $\int_{-\infty}^{\infty} \lambda \, d\, e_\lambda$, where e_λ's are* sa-*idempotents and $e_\lambda \uparrow$, in the sense that $l_x = \int_{-\infty}^{\infty} \lambda \, d\, E_\lambda$, where E_λ is the spectral decomposition of l_x and e_λ is such that $E_\lambda = l_{e_\lambda}$.*

We will prove the following theorem which will be useful later on.

THEOREM 3. *Let $w_1, w_2, \ldots, w_n, \ldots$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} w_n^2$ is convergent, and an infinity of w_n's are different from zero. Then there exists a sequence of positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ such that* (i) $\sum_{n=1}^{\infty} \alpha_n w_n^2 < \infty$ *and* (ii) $\sum_{n=1}^{\infty} \alpha_n^2 w_n^2$ *diverges.*

PROOF. Without loss of generality we will assume that no $w_n = 0$. There exists a sequence $n_1 < n_2 < \ldots < n_r \ldots$ of integers n_r tending to ∞ such that $w_{n_r}^2 = \epsilon_r / 2^r$, where $0 < \epsilon_r \leqslant 1$ for each r. Put $\alpha_n = 0$ if n is not any of the numbers n_r and $\alpha_{n_r} = \sqrt{(2^r/\epsilon_r)}$. Then $\sum_{n=1}^{\infty} \alpha_n w_n^2 = \sum_{n=1}^{\infty} \sqrt{(\epsilon_r/2^r)}$ which converges and $\sum_{n=1}^{\infty} \alpha_n^2 w_n^2 = \sum_{n=1}^{\infty} 1$ which diverges.

Now we will see that the centre is a finite dimensional algebra over C. We note that the centre Z of H is also an H-algebra.

THEOREM 4. *Let H be a commutative H-algebra which has no proper closed ideal I. Then H is isomorphic to the field C of complex numbers. If $\|1\| = 1$ then the norm in H corresponds to the modulus of a number z in C. The involution* corresponds to the complex conjugate.*

PROOF. First of all we see that there is no sa-idempotent other than 1 in H from Theorem 1.

Hence if x is a self-adjoint element of H then, the formula $x = \int_{-\infty}^{\infty} \lambda\, de_\lambda$ gives $x = \lambda_0.1$ for some real number λ_0.

Now any element $Z \in H$ can be expressed as $x + iy$, where x and y are self-adjoint elements of H. Hence $z = (\lambda + i\,\mu)\ 1$, where λ and μ are real. Thus we find that H is algebraically isomorphic to C. Now, if $z = (\lambda + i\,\mu).1$ then obviously $z^* = (\lambda - i\,\mu).\ 1$. Hence the involution corresponds to taking the complex conjugate in C. Now if $z = (\lambda + i\,\mu).\ 1$, then $\|z\| = |\lambda + i\,\mu|\ \|1\|$.

Hence if $\|1\| = 1$ then $\|z\| = |\lambda + i\,\mu|$ or H is also isometric with C. Otherwise the norm is only a constant multiple of the modulus.

This completes the proof of the theorem.

THEOREM 5. *Let $e_1, e_2, \ldots, e_n$ be any sequence of self-adjoint orthogonal idempotents. Then $\sum_{n=1}^{\infty} \|e_n\|^2$ is convergent. Let $\|e_n\| = \omega_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be any sequence of real numbers such that $\sum_{n=1}^{\infty} \alpha_n^2\, \omega_n^2$ is convergent. Then $\sum_{n=1}^{\infty} \alpha_n\, e_n$ converges in norm to some element in H.*

PROOF. We know that $\{1-(e_1 + e_2 + \ldots + e_n)\}$ is a sa-idempotent and that

$$\|(e_1 + e_2 + \ldots + e_n)\|^2 + \|\{1 - (e_1 + e_2 + \ldots + e_n)\}\|^2 = \|1\|^2.$$

Hence $\|(e_1 + e_2 + \ldots + e_n)\|^2 \leqslant \|1\|^2$ for all n. Hence

$$\sum_{m=1}^{\infty} \|e_m\|^2 = \|(e_1 + e_2 + \ldots + e_n)\|^2 \leqslant \|1\|^2$$

and so $\Sigma\ \|e_n\|^2$ is convergent.

The latter part of the theorem is obvious.

THEOREM 6. *Let H be a commutative H-algebra with a unit 1. Then H is the direct sum of a finite number of H-algebras, each of which is isomorphic to C.*

PROOF. First of all we will see that there cannot be an infinite set of orthogonal sa-idempotents. For, if possible let $e_1, e_2, \ldots, e_n, \ldots$ be any sequence of orthogonal sa-idempotents. Then $\sum_{n=1}^{\infty} \|e_n\|^2$ is convergent. Let us put $\omega_n = \|e_n\|$ for $n = 1, 2, \ldots$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be any sequence of real numbers such that $\sum_{n=1}^{\infty} \lambda_n^2 \omega_n^2$ is convergent. Then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges. Let $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Then $x^2 = \lim_{m\to\infty} x \sum_{n=1}^{m} \lambda_n e_n$ since l_x is continuous in H. Hence $x^2 = \sum_{n=1}^{\infty} \lambda_n^2 e_n$ and $\sum_{n=1}^{\infty} \lambda_n^4 \omega_n^2 = \|x\|^2$. Hence the convergence of $\sum_{n=1}^{\infty} \lambda_n^2 \omega_n^2$ implies that $\sum_{n=1}^{\infty} \lambda_n^4 \omega_n^2$ is convergent whatever the sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ may be. But this is impossible from Theorem 1. Hence every set of orthogonal sa-idempotents is finite.

From this we see that every descending sequence $e_1 \geqslant e_2 \geqslant \ldots$ of sa-idempotents should be finite. Otherwise, we can get an infinite set of orthogonal sa-idempotents.

Hence we see that there are minimal sa-idempotents, i.e. sa-idempotents e such that $e_1 \leqslant e$ and e_1 is a sa-idempotent together imply that $e_1 = e$.

Let e_1 and e_2 be two sa-idempotents which are minimal. If $e_1 e_2 \neq 0$ then $e_1 e_2$ will be a sa-idempotent $\leqslant e_1$ and also e_2 which is impossible since e_1 and e_2 are minimal. Hence $e_1 e_2 = 0$ or equivalently e_1 and e_2 are orthogonal. Hence the set of minimal sa-idempotents of H can only be finite. Let us denote by $e_1, e_2, \ldots, e_n$ all the minimal sa-idempotents.

Now $e_i H$ and $e_j H$ are two orthogonal closed ideals of H which are also H-algebras by themselves. Consider $H_1 = H \ominus \{e_1 H \oplus e_2 H \oplus \ldots \oplus e_n H\}$. If $H_1 \neq 0$, then H_1 must contain a sa-idempotent and hence a minimal sa-idempotent different from $e_1, e_2, \ldots, e_n$. Hence $H_1 = 0$ and $H = e_1 H \oplus e_2 H \oplus e_3 H \oplus \ldots \oplus e_n H$. Now each $e_i H$ has only one sa-idempotent e_i since e_i is minimal and is the

identity of $e_i H$. Hence each $e_i H$ has no proper closed ideal by Theorem 2. Hence each $e_i H$ is isomorphic to C by Theorem 4.

THEOREM 7. *Let H be any H-algebra with an identity* 1 *and such that* 1 *is the only central* sa-*idempotent. Then H is isomorphic to the algebra of all matrices of order n led by the trace topology and the involution * corresponds to taking the adjoint of a matrix. If $\|1\| = K$ and $x \in H$ corresponds to the matrix T then $\|x\|^2 = (K/n)\,\sigma(TT^*)$, where $\sigma(T)$ is the trace of the matrix T and T^* is the adjoint of T.*

PROOF. As before we can show that there are minimal sa-idempotents e_i and an orthogonal set of such idempotents are finite in number. If $e_1, e_2, \dots, e_n$ is one such set of minimal sa-idempotents then

$$1 = e_1 + e_2 + \dots + e_n \text{ and } H = \bigoplus_{i,j=1}^{n} (e_i H e_j).$$

Now we have that $e_i H e_i$ is an H-algebra with a unit and having no other sa-idempotent since e_i is minimal. Hence $e_i H e_i$ is one dimensional (which can be proved by an argument similar to that of Theorem 4). By following standard reasoning as in [3], we see that $e_i H e_j$ is one dimensional for each i and j. Now we will choose n^2 elements e_{ij} such that $e_{ii} = e_i$ for all $i = 1, 2, \dots, n$ and $e_{ji} = e^*_{ij}$ for $i \leqslant j$ $(i, j = 1, 2, \dots, n)$. Then we see that every element $x \in H$ is $= \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}$, where $\lambda_{ij} \in C$ and is uniquely fixed by x. Thus to each x there corresponds a matrix

$$T = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \cdots & \cdots & \cdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{bmatrix}$$

of order n. We can verify that this is an algebraic isomorphism from H onto the set of all matrices of order n. It is also easily verified that $\|x\|^2 = (K/n)\,\sigma(TT^*)$.

COROLLARY. *Let H be any H-algebra with a unit. Then H is isomorphic to the direct sum of its minimal closed two-sided ideals*

H_i which can be only finite in number. Each ideal has only one central sa-idempotent e_i. Thus H is isomorphic to the direct product of a finite number of matrix algebras M_i of finite order and led by the trace topology.

PROOF. Let Z be the centre of H. Then $Z = C_1 \oplus C_2, \oplus \ldots \oplus C_n$, where each C_i is isomorphic to the field C of complex numbers, (from Theorem 6). Hence each C_i has only one sa-idempotent e_i which is its identity. Obviously $H = e_1 H \oplus \ldots \oplus e_i H \ldots \oplus e_n H$, where $e_i H$ is a minimal closed two-sided ideal having only one central sa-idempotent e_i. Now from Theorem 7 each $e_i H$ is isomorphic to the matrix algebra of some order n_i led by the trace topology. Hence the result.

THEOREM 8. *Let H be an H-algebra with a unit 1. Then there exists a constant K such that $\|xy\| \leqslant K \|x\| \|y\|$ for all $x, y \in H$.*

PROOF. This follows from the structure Theorem 7.

THEOREM 9. *Let A be an H-algebra such that $\|xy\| \leqslant K \|x\| \|y\|$ for all $x, y \in A$ and a fixed number K. Then A can be given a new norm $[[x]]$ so that it remains to be an H-algebra even in the new norm and $[[xy]] \leqslant [[x]][[y]]$, for all $x, y \in A$ and the new norm is equivalent to the old norm in A.*

Proof. The norm $[[x]]$ defined by $[[x]] = K\|x\|$ for any $x \in A$ satisfies the conditions of the theorem.

REMARKS. Let A be an algebra with a unit and also an inner product space in which l_a and r_a are continuous for all $a \in A$. Similar systems are considered in [6], [5] and [2]. A need not be complete. Let $\bar{A}$ be the completed space of A. Then, it follows from the continuity of $\|x\|$ and the theorems proved above, that $\bar{A}$ can be made to be an algebraic extension of A and become an H-algebra if and only if, there exists a constant K such that $\|xy\| \leqslant K \|x\| \|y\|$ for all $x, y \in A$. Now in this case, l_x and r_x have the same bound as given under Remark 1. Let us denote by $|||a|||$ this common bound of l_a and r_a. Then the existence of such a K is equivalent

to saying that the set E, where $E = \{\lambda : \lambda = |||x|||_{x \in A \& \|x\| = 1}\}$ is a bounded set. If E is bounded, then K = upper bound of E. Thus we have

THEOREM 10. *Let A be an algebra with a unit, and also an inner product space under the norm of which l_a and r_a are continuous for all $a \in A$. Then the completion $\bar{A}$ can be made an algebraic extension of A, by continuity if and only if the set of all number of the form $|||x|||$, where $x \in A$ and $\|x\| = 1$ is a bounded set or equivalently, there exists a number K, such that $\|x\, y\| \leqslant K\, \|x\|\, \|y\|$ for all $x, y \in A$.*

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CONVERGENCE THEOREMS AND ASYMPTOTIC PROPERTIES OF A GENERALIZED STIELTJES TRANSFORM

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1. Introduction. The Stieltjes transform arises as an iteration of the Laplace transform, i.e. if we take

$$f_0(s) = \int_0^\infty e^{-st} \psi(t)\, dt,$$

where

$$\psi(s) = \int_0^\infty e^{-st} \phi(t)\, dt$$

then

$$f_0(s) = \int_0^\infty \frac{\phi(t)\, dt}{s+t}, \tag{1}$$

and (1) is referred to as the Stieltjes transform. On replacing $\phi(t)dt$ by $d\alpha(t)$ we get the more general case of (1) in the form

$$f_0(s) = \int_0^\infty \frac{d\alpha(t)}{s+t}. \tag{2}$$

Varma [2] has given a generalization of Laplace integral in the form

$$f(s) = \int_0^\infty (st)^{m-\frac{1}{2}} e^{-\frac{1}{2}st} W_{k,m}(st)\, \psi(t)\, dt. \tag{3}$$

He has also shown that if we take $f(s)$ to be the transform of $\psi_1(t)$ in this generalized sense and $\psi_1(s)$ to be the ordinary Laplace transform of $\phi(t)$, we get

$$f(s) = \frac{1}{s} \cdot \frac{\Gamma(2m+1)}{\Gamma(m-k+3/2)} \int_0^\infty F\left[\begin{matrix} 2m+1,\, 1 \\ m-k+3/2 \end{matrix};\, -\frac{t}{s}\right] \phi(t)\, dt, \tag{4}$$

where $\mathrm{Re}(2m + 1) > 0$. If $k + m = \frac{1}{2}$, the hypergeometric function degenerates into a binomial expression and we get the ordinary Stieltjes transform. If $k - m = \frac{1}{2}$, and $2m + 1 = \rho$, then also the hypergeometric function degenerates into another binomial expression and we get

$$\theta(s) = \frac{1}{\Gamma(\rho)} s^{-\rho+1} f(s) = \int_0^\infty \frac{\phi(t)\, dt}{(s + t)^\rho}, \tag{5}$$

which is a generalized form of Stieltjes transform.

Further Varma has shown that if $f_1(s)$ be the ordinary Laplace transform of $\psi_2(t)$ and $\psi_2(s)$ be the generalized Laplace transform of $\phi(t)$, we have

$$f_1(s) = \frac{\Gamma(2m + 1)}{\Gamma(m - k + 3/2)} \int_0^\infty \frac{1}{t} F\left[\begin{matrix} 2m + 1, 1 \\ m - k + 3/2 \end{matrix}; -\frac{s}{t}\right] \phi(t)\, dt. \tag{6}$$

If $k + m = \frac{1}{2}$ in (6), we get the Stieltjes transform.

By a suitable change of variables (6) may be reduced to (4) and vice versa.

For convenience we will denote

$$F\left[\begin{matrix} 2m + 1, 1 \\ m - k + 3/2 \end{matrix}; -\frac{t}{s}\right] \quad \text{and} \quad \frac{\Gamma(2m + 1)}{\Gamma(m-k + 3/2)}$$

by $F[t, s]$ and A respectively in this paper.

We shall usually be concerned with the more general case in which (4) is replaced by a Stieltjes integral

$$f(s) = A s^{-1} \int_0^\infty F[t, s]\, d\alpha(t). \tag{7}$$

Let $\alpha(t)$ be of bounded variation in the interval $0 \leqslant t \leqslant R$ for every positive R. We now define the improper integral (7) as

$$\lim_{R\to\infty} A s^{-1} \int_0^R F[t, s]\, d\alpha(t)$$

for any complex $s(= \sigma + i\tau)$. If this limit exists for a given value of s we say (7) converges for that value of s.

If $\alpha(t)$ is of bounded variation in the interval $\epsilon \leqslant t \leqslant R$ for every positive ϵ and any positive R, then we use the notation

$$A\, s^{-1} \int_{0+}^{\infty} F[t, s]\, d\alpha(t) = \lim_{\epsilon \to 0,\, R \to \infty} A\, s^{-1} \int_{\epsilon}^{R} F[t, s]\, d\alpha(t). \tag{8}$$

In this paper we will give some theorems for the transforms (7) and (8). The results deduced give known results for (2) when $k + m = \frac{1}{2}$ [3]. Further if $2m + 1 = \rho$ and $k - m = \frac{1}{2}$, we get corresponding results for (5).

2. Convergence theorems.

We prove

THEOREM 2.1. *If the integral* (7) *converges for a point* $s = s_0$, *not on the negative real axis,* $\tau = 0$, $\sigma \leqslant 0$ *then it converges for every such point, provided that* Re $(2m + 1) > 0$ *and* $m - k + 3/2 \neq 0, -1, -2, \ldots$.

PROOF : Let us set

$$\beta(t) = A\, s_0^{-1} \int_{0}^{t} F[t, s]\, d\alpha(t) \quad (0 \leqslant t < \infty).$$

Then for any s not on the negative real axis and for any positive R

$$A\, s^{-1} \int_{0}^{R} F[t, s]\, d\alpha(t) = s_0\, s^{-1} \beta(R)\, [F[R, s]/F[R, s_0]] -$$

$$- s_0\, s^{-1} \int_{0}^{R} (d/dt) [\, F[t, s]/F[t, s_0]\,]\, \beta(t)\, dt.$$

If $2m$ is not an integer or zero ([1], p. 63, (17)) we have

$$F[t, s] \sim \frac{\Gamma(m-k+3/2)\Gamma(-2m)}{\Gamma(-m-k+\frac{1}{2})}\, s^{2m+1}\, t^{-2m-1} +$$

$$+ \frac{m-k+\frac{1}{2}}{2m}\, s t^{-1} \quad (t \to \infty). \tag{9}$$

Also if $2m$ is zero or a positive integer and $m - k + \frac{1}{2} \neq 0$ or a positive integer, we have ([1], p. 63, (18))

$$AF[t, s] \sim (-1)^{2m}\Gamma^{-1}(-m-k+\tfrac{1}{2})\, s^{2m+1}\, t^{-2m-1}\, [\log(t/s)+h_0] + \\ + \frac{\Gamma(2m)}{\Gamma(m-k+\frac{1}{2})}\, st^{-1} \quad (t \to \infty), \qquad (10)$$

the last term being zero when $2m = 0$.

Further, if $2m$ is zero or a positive integer and $-m-k+\frac{1}{2}$ is zero or a positive integer, we have ([1], p. 63, (19))

$$AF[t, s] \\ \sim (-1)^{m-k+\frac{1}{2}}\, s^{m-k+3/2}\, t^{k-m-3/2} + \\ + (-1)^{2m}\Gamma^{-1}(-m-k+\tfrac{1}{2})\, s^{2m+1}\, t^{-2m-1}\, [\log(t/s)+h_0] + \\ + \frac{(2m-1)!}{(m-k-\frac{1}{2})!}\, st^{-1} \quad (t \to \infty) \qquad (11)$$

$$\left(h_0 = \psi(1) - \psi(\tfrac{1}{2}-m-k) \text{ and } \psi(z) = \frac{d}{dz}\log\Gamma(z)\right),$$

the second and third terms being zero if $-m-k+\frac{1}{2} = 0$ or $2m = 0$.

Further, we have ([1], p. 105, (2))

$$F[t, s] = \left(1+\frac{t}{s}\right)^{-m-k-\frac{1}{2}} F\left[\begin{matrix} -m-k+1/2,\, m-k+1/2 \\ m-k+3/2 \end{matrix}; -\frac{t}{s}\right]. \qquad (12)$$

Using the results (9), (10), (11), and (12), we have

$$\{F[R, s]/F[R, s_0]\} \sim s\, s_0^{-1} \qquad (R \to \infty),$$

if (i) $2m$ is zero or positive integer, or

(ii) $k + m = \frac{1}{2}$, or

(iii) $2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$ and $\operatorname{Re} m > 0$; and

$$\sim (s/s_0)^{2m+1} \quad (R \to \infty);$$

if (i) $k - m = \frac{1}{2}$, or

(ii) $2m \neq 0$ or a positive integer, and $\operatorname{Re} m < 0$.

Thus

$$s_0 s^{-1} \beta(R)\{F[R, s]/F[R, s_0]\}$$

tends to a finite limit as R tends to infinity.

The integral

$$U \equiv \int_0^\infty \frac{d}{dt}\{F[t, s]/F[t, s_0]\}\, \beta(t)\, dt$$

is absolutely convergent, since

$$|U| \leqslant M \left| \int_0^\infty \frac{d}{dt}\{F[t,s]/F[t,s_0]\}\, dt \right|$$

$$\leqslant M \left| \lim_{t\to\infty} \{F[t, s]/F[t,s_0]\} - 1 \right|,$$

where M is the upper bound of $|\beta(t)|$ in $0 \leqslant t < \infty$.

Now we have seen that

$$F[t, s]/F[t, s_0]$$

tends to (s/s_0) or $(s/s_0)^{2m+1}$ as $t \to \infty$, which are finite and therefore the above integral is absolutely convergent. Hence the theorem is established.

THEOREM 2.2. *If the integral* (7) *converges, it converges uniformly in any closed bounded region not containing a point of the negative real axis,* $\tau = 0$, $\sigma \leqslant 0$.

PROOF. Let us set $\beta(t)$ as in Theorem 2.1 and let us determine R so large that for $R \leqslant t < \infty$,

$$|\beta(t) - \beta(\infty)| < \epsilon.$$

Therefore

$$V \equiv \left| A\, s^{-1} \int_R^\infty F[t, s]\, d\alpha(t) \right|$$

$$\leqslant |f(s_0) - \beta(R) s_0 s^{-1} \{F[R, s]/F[R, s_0]\}| +$$
$$+ (|\beta(\infty)| + \epsilon)\, |1 - s_0 s^{-1}\{F[R,s]/F[R,s_0]\}| \tag{13}$$

or

$$V \leqslant \left| \left(\frac{s}{s_0}\right)^{2m} \beta(\infty) - \beta(R)\, s_0 s^{-1}\{F[R,s]/F[R,s_0]\} \right| +$$
$$+ (|\beta(\infty)| + \epsilon)\, |(s s_0^{-1})^{2m} - s_0 s^{-1}\{F[R,s]/F[R,s_0]\}|. \tag{14}$$

Also $F[R, s]/F[R, s_0]$ tends uniformly to (s/s_0) or $(s/s_0)^{2m+1}$ as $R \to \infty$ in a closed bounded region not containing a point of the negative real axis. In both cases we see that the right hand sides of (13) and (14) tend to zero as R tends to infinity and the theorem is proved.

Corollary 2.2a. *If the integral* (7) *converges, it represents an analytic single valued function in the complex plane cut along the negative real axis.*

Corollary 2.2b. *If the integral* (7) *converges to* $f(s)$, *then for any* s *not on the negative real axis*

$$f^{(n)}(s) = (-1)^n A\Gamma(n+1)\, s^{-n-1} \times$$
$$\times \int_0^\infty F\left[2m+1, n+1; m-k+\frac{3}{2}; -\frac{t}{s}\right] d\alpha(t). \qquad (15)$$

For ([1], p. 102, (21))

$$\frac{d}{dz}\,[z^a F(a, b; c; z)] = az^{a-1} F(a+1, b; c; z).$$

Whence

$$\frac{d}{ds}\,[s^{-1} F[t, s]] = (-1)\, s^{-2}\, F\left[2m+1, 2; m-k+\frac{3}{2}; -\frac{t}{s}\right].$$

Hence differentiating repeatedly with respect to s we have

$$\frac{d^n}{ds^n}\,[s^{-1} F[t, s]] = (-1)^n \Gamma(n+1)\, s^{-n-1} \times$$
$$\times F\left[2m+1, n+1; m-k+\frac{3}{2}; -\frac{t}{s}\right],$$

from which the result (15) follows.

3. Order properties of $\alpha(t)$**.** We now proceed from the convergence properties of the integral to the order properties of $\alpha(t)$.

Theorem 3.1. *If either of the integrals* (7) *or* (8) *converges, then provided that* Re $(2m+1) > 0$, $m - k + 3/2 \neq 0, -1, -2, \ldots$

$$\alpha(t) = o(t) \qquad (t \to \infty) \qquad (16a)$$

if (i) $2m$ *is a positive integer, or*

(ii) $2m \neq 0$ *or a positive integer,* $k \pm m \neq \frac{1}{2}$ *and* $\operatorname{Re} m \geqslant 0$, *or*

(iii) $k + m = \frac{1}{2}$;

$$\alpha(t) = o(t^{2m+1}) \quad (t \to \infty); \tag{16b}$$

if (i) $k - m = \frac{1}{2}$, *or*

(ii) $2m \neq 0$ *or a positive integer and* $\operatorname{Re} m < 0$; *and*

$$\alpha(t) = o(t/\log t) \quad (t \to \infty); \tag{16c}$$

if $2m = 0$ *and* $-k + \frac{1}{2} \neq 0$.

PROOF. Let (7) or (8) converge at $s = s_0$ on the positive real axis. Let us set

$$\beta(t) = A\, s_0^{-1} \int_{s_0}^{t} F[t, s_0]\, d\alpha(t).$$

Then

$$\alpha(t) - \alpha(s_0) = \int_{s}^{t} d\alpha(u)$$
$$= s_0 A^{-1} [\beta(u) \{F[u, s_0]\}^{-1}]_{s_0}^{t} -$$
$$- s_0 A^{-1} \int_{s_0}^{t} \beta(u) \frac{d}{du} \{F[u, s_0]\}^{-1} du,$$

or

$$A\, s_0^{-1} [\alpha(t) - \alpha(s_0)] F[t, s_0]$$
$$= \beta(t) - F[t, s_0] \int_{s_0}^{t} \beta(u) \frac{d}{du} \{F[u, s_0]\}^{-1} du.$$

By using the mean value theorem of integral calculus, we have

$$A\, s_0^{-1} [\alpha(t) - \alpha(s_0)]\, F[t, s_0] \sim \beta(\infty) - \beta(\infty) \quad (t \to \infty),$$

since $F[t, s_0]$ approaches zero as t tends to infinity in all cases mentioned in the theorem.

We can show that the limit of $\alpha(t)/t$ or $\alpha(t)/t^{2m+1}$ or $\alpha(t)/(t/\log t)$ is zero under the conditions for (16a), (16b) and (16c) respectively.

THEOREM 3.2. *If either of the integrals* (7) *or* (8) *converges, then* $\alpha(0+)$ *exists.*

The proof is similar to that of Theorem 3.1.

THEOREM 3.3. *If for some positive* δ

$$\alpha(t) = O(t^{1-\delta}) \qquad (t \to \infty) \tag{17a}$$

and (i) $2m$ *is a positive integer, or*

(ii) $2m \neq 0$ *or a positive integer,* $k \pm m \neq \frac{1}{2}$ *and* Re $m \geqslant 0$; *or*

(iii) $k + m = \frac{1}{2}$;

or if

$$\alpha(t) = O(t^{2m+1-\delta}) \qquad (t \to \infty) \tag{17b}$$

and (i) $k - m = \frac{1}{2}$, $k + m \neq \frac{1}{2}$, *or*

(ii) $2m \neq 0$, *or a positive integer,* $k \pm m \neq \frac{1}{2}$, *and* Re $m < 0$;

or if

$$\alpha(t) = O(t^{1-\delta}/\log t) \qquad (t \to \infty) \tag{17c}$$

and $2m = 0$ *and* $-k + \frac{1}{2} \neq 0$;

then (7) *converges, provided that* Re $(2m + 1) > 0$, *and* $m - k + (3/2) \neq 0, -1, -2, \ldots$.

If in addition $\alpha(0+)$ *exists, then* (8) *converges.*

PROOF. Since $\alpha(t)$ is of bounded variation in $0 \leqslant t \leqslant R$ for every $R > 0$, we have

$$A s^{-1} \int_0^R F[t, s]\, d\,\alpha(t)$$

$$= A s^{-1}[\alpha(R) F[R, s] - \alpha(0)] + \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \times$$

$$\times\, s^{-2} \int_0^R F[2m+2, 2\,; m-k+5/2\,; -t/s]\, \alpha(t)\, dt.$$

In all cases mentioned in the theorem

$$\alpha(R)\, F[R, s] = O(R^{-\delta}) \quad (R \to \infty).$$

Also we have

$$\alpha(t)\, F[2m+2,\, 2\,;\, m-k+5/2,\, -t/s] = O(t^{-1-\delta}) \quad (t \to \infty).$$

Hence it is clear that

$$A\, s^{-1} \int_0^R F[t, s]\, d\alpha(t)$$

approaches a finite limit as R becomes infinite.

For the integral (8) a similar proof holds.

4. Asymptotic properties of the transform. We will next study the asymptotic behaviour of the transforms (7) and (8) as s becomes infinite or approaches zero, where s is real and positive.

THEOREM 4.1a. *If $f(s)$ is defined by the convergent integral* (7) *with* $\alpha(0) = 0$ *and*

(i) $\mathrm{Re}\,(2m+1) > 0$,

(ii) $m - k + 3/2 \neq 0, -1, -2, \ldots,$ *and*

(iii) (a) *$2m$ is zero or a positive integer, or*

(b) $2m \neq 0$ *or a positive integer,* $k \pm m \neq \frac{1}{2}$ *and* $\mathrm{Re}\; m \geqslant 0$ *or*

(c) $k + m = \frac{1}{2}$,

then

$$f(s) \sim A\, \alpha(0+)s^{-1}, \quad (s \to 0+) \tag{18}$$

$$f(s) \sim o(1), \quad (s \to \infty). \tag{19}$$

PROOF. Let us set $\beta(t)$ as in Theorem 2.1. Then

$$f(s) = s_0 s^{-1} \left[\frac{F\,[t, s]}{F\,[t, s_0]}\, \beta\,(t) \right]_0^\infty - s_0 s^{-1} \int_0^\infty \frac{d}{dt} \left\{ \frac{F\,[t, s]}{F\,[t, s_0]} \right\} \beta\,(t)\, dt$$

Therefore

$$\lim_{s \to 0+} s f(s) = \lim_{s \to 0+} (-1)\, s_0 \int_0^\infty \frac{d}{dt} \left\{ \frac{F\,[t, s]}{F\,[t, s_0]} \right\} \beta\,(t)\, dt.$$

Now

$$s_0 \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} dt = s - s_0.$$

Also

$$\beta(t) = As_0^{-1} F[t_1, s_0]\, \alpha(t) \quad 0 < t_1 < t.$$

Therefore

$$\beta(0+) = As_0^{-1}\, \alpha(0+).$$

Hence

$$P \equiv \left| \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} \beta(t)\, dt - \beta(0+)(s s_0^{-1} - 1) \right|$$

$$= \left| \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} (\beta(t) - \beta(0+))\, dt \right|.$$

Given ϵ, we choose δ so small that for $0 \leqslant t \leqslant \delta$

$$|\beta(t) - \beta(0+)| < \epsilon,$$

and let M be the upper bound of $|\beta(t) - \beta(0+)|$ for all positive t.

Then

$$|P| \leqslant \left| \int_0^\delta \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} |\beta(t) - \beta(0+)|\, dt \right| +$$

$$+ \left| \int_\delta^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} |\beta(t) - \beta(0+)|\, dt \right|$$

$$\leqslant \epsilon \left| \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} dt \right| + M \left| \int_\delta^\infty \frac{d}{dt} \left\{ \frac{F[t,s]}{F[t,s_0]} \right\} dt \right|$$

$$\leqslant \epsilon\, |s s_0^{-1} - 1| + M\, |s s_0^{-1} - \{F[\delta, s]/F[\delta, s_0]\}|.$$

Hence $\limsup_{s\to 0} P = 0$, since ϵ is arbitrary. Therefore

$$\lim_{s\to 0+} sf(s) = \lim_{s\to 0+} (-s + s_0)\, \beta(0+) = A\, \alpha(0+),$$

or (18) is established.

Again, let

$$Q \equiv \left| s^{-1} \int_0^\infty \left\{ \frac{d}{dt} \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt - \beta(\infty)\,(1 - s_0\, s^{-1}) \right|$$

$$= \left| s^{-1} \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} (\beta(t) - \beta(\infty))\, dt \right|.$$

Given ϵ' we choose R so large that for $R \leqslant t < \infty$

$$|\beta(t) - \beta(\infty)| < \epsilon',$$

and let M' be the upper bound of $|\beta(t) - \beta(\infty)|$ for all positive t. Then

$$Q \leqslant \left| s^{-1} \epsilon' \int_R^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} dt \right| + M' \left| s^{-1} \int_0^R \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} dt \right|$$

$$\leqslant \epsilon' \,|1 - s_0 s^{-1}| + M' \left| s^{-1} \left\{ \left(\frac{F[R, s]}{F[R, s_0]} \right) - 1 \right\} \right|.$$

Therefore $\overline{\lim_{s \to \infty}}\; Q = 0$, since ϵ' is arbitrary. Hence

$$\lim_{s \to \infty} f(s) = \lim_{s \to \infty} [f(s_0) - \beta(\infty)\,(1 - s_0\, s^{-1})] = 0.$$

THEOREM 4.1b. *If $f(s)$ is defined by convergent integral* (7) *with* $\alpha(0) = 0$ *and*

(i) $\mathrm{Re}(2m + 1) > 0$,

(ii) $m - k + \frac{3}{2} \neq 0, -1, -2, \ldots$ *and*

(iii) (a) *$2m$ is a positive integer, or*

(b) *$2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$,* $\mathrm{Re}\, m > 0$, *or*

(c) $k + m = \frac{1}{2}$;

then

$$f^{(n)}(s) \sim (-1)^n n!\, A\, \alpha(0+)\, s^{-n-1} \quad (s \to 0+;\ n = 1, 2, \ldots) \qquad (20)$$

$$f^{(n)}(s) = o(s^{-n}) \quad (s \to \infty;\ n = 1, 2, \ldots). \qquad (21)$$

PROOF. Integrating by parts we have

$$f(s) = \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \frac{1}{s^2} \int_0^\infty F\left[\begin{matrix} 2m+2, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] \alpha(t)\, dt.$$

But ([1], p. 21)

$$\frac{d}{ds}\left\{\frac{1}{s^2} F\left[\begin{matrix} 2m+2, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right]\right\} = (-1).2s^{-3} F\left[\begin{matrix} 2m+2, 3 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right].$$

So

$$\frac{d^n}{ds^n}\left\{\frac{1}{s^2} F\left[\begin{matrix} 2m+2, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right]\right\} = (-1)^n (n+1)!\, s^{-n-2} \times$$

$$\times F\left[\begin{matrix} 2m+2, n+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right],$$

and

$$f^{(n)}(s) = (-1)^n \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \cdot (n+1)! \times$$

$$\times s^{-n-2} \int_0^\infty F\left[\begin{matrix} 2m+2, n+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] \alpha(t)\, dt. \qquad (22)$$

The above integral converges absolutely by Theorem 3.1.

Therefore

$$I \equiv A^{-1}(-1)^n f^{(n)}(s) \frac{s^{n+1}}{n!} - \alpha(0+),$$

$$= \frac{(n+1)(2m+1)}{s(m-k+3/2)} \int_0^\infty F\left[\begin{matrix} 2m+2, n+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] \times$$

$$\times \{\alpha(t) - \alpha(0+)\}\, dt,$$

since ([1] p. 79 (4))

$$\int_0^\infty F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt = \frac{s(m-k+3/2)}{(n+1)(2m+1)},$$

provided that $m-k+5/2 \neq 0, -1, -2, \ldots.$

Given $\epsilon > 0$, we choose δ so small that for $0 \leqslant t \leqslant \delta$

$$|\alpha(t) - \alpha(0+)| < \epsilon,$$

and let
$$M = \underset{\delta \leqslant t < \infty}{\text{l.u.b.}} |\alpha(t) - \alpha(0+)| t^{-1},$$

where M is finite by Theorem 3.1. Then

$$|I| \leqslant \epsilon \left| \frac{(n+1)(2m+1)}{s(m-k+3/2)} \int_0^\delta F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right| +$$

$$+ M \left| \frac{(n+1)(2m+1)}{s(m-k+3/2)} \int_\delta^\infty t\ F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right|$$

$$\leqslant \epsilon \left| \frac{(n+1)(2m+1)}{s(m-k+3/2)} \int_0^\infty F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right| +$$

$$+ M \left| \frac{(n+1)(2m+1)}{s(m-k+3/2)} \int_0^\infty t\ F\left[\begin{matrix} n+2, 2m+2 \\ m-k+3/2 \end{matrix}; -\frac{t}{s}\right] dt\right.$$

$$\leqslant \epsilon + M \left| \frac{s(m-k+\frac{1}{2})}{2mn} \right|,$$

or

$$\leqslant \epsilon + M \left| \frac{s}{n} \right| \text{ if } k+m = \tfrac{1}{2}.$$

Hence $\lim\limits_{s \to 0+} I = 0$, since ϵ is arbitrary.

Hence (20) is established.

Again, using Theorem 3.1, determine R so large that for $R \leqslant t < \infty$, $|\alpha(t)| < \epsilon' t$.

From (22) we have

$$s^n f^{(n)}(s) = (-1)^n (n+1)! \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \times$$

$$\times \frac{1}{s^2} \int_0^\infty F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] \alpha(t)\, dt,$$

or

$$\left| s^n f^{(n)}(s) \right|$$

$$\leqslant \left| s^{-2}(n+1)! \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \times \right.$$

$$\left. \times \int_0^R F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] |\alpha(t)|\, dt \right| +$$

$$+ \left| s^{-2}(n+1)! \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \epsilon' \int_R^\infty t\, F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right|,$$

$$\leqslant \left| s^{-2}(n+1)! \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} N \int_0^\infty F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right| +$$

$$+ \left| s^{-2}(n+1)! \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \epsilon' \int_0^\infty t\, F\left[\begin{matrix} n+2, 2m+2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt \right|$$

(where $N = \underset{0 \leqslant t \leqslant R}{\text{l.u.b.}} |\alpha(t)|$)

$$\leqslant |s^{-1} n!\, A\, N| + \epsilon' \left| \frac{\Gamma(n+1)\,\Gamma(2m)}{\Gamma(m-k+\frac{1}{2})} \right|, \quad \text{or}$$

$$\leqslant |s^{-1} n!\, N| + \epsilon'\, \Gamma(n+1), \text{ if } k+m = \tfrac{1}{2}.$$

Hence $\limsup_{s \to \infty} s^n f^{(n)}(s) = 0.$

THEOREM 4.2a. *If $f(s)$ is defined by convergent integral* (7) *with* $\alpha(0) = 0$, *and*

(1) Re $(2m+1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots$

(3) (i) $k - m = \frac{1}{2}$, *or*

(ii) $2m \neq 0$ *or a positive integer*, $k \pm m \neq \frac{1}{2}$, Re $(m) < 0$,

then

$$f(s) \sim A\, \alpha(0+)\, s^{-1} \quad (s \to 0+), \tag{23}$$

$$s^{-2m} f(s) = o(1) \quad (s \to \infty). \tag{24}$$

PROOF. Following the arguments of Theorem 4.1a, we have

$$sf(s) = s^{2m+1} s_0^{-2m} f(s_0) - s_0 \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt.$$

So

$$\lim_{s \to 0+} sf(s) = \lim_{s \to 0+} (- s_0) \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt.$$

Now

$$\int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} dt = (s\, s_0^{-1})^{2m+1} - 1.$$

Hence

$$P' = \left| \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt - \beta(0+) \{(s\, s_0^{-1})^{2m+1} - 1\} \right|,$$

$$\leqslant \epsilon \,|\, (s\, s_0^{-1})^{2m+1} - 1 \,| + M \left| (s\, s_0^{-1})^{2m+1} - \frac{F[\delta, s]}{F[\delta, s_0]} \right|,$$

and therefore $\limsup_{s \to 0+} P' = 0$, since ϵ is arbitrary.

Therefore

$$\limsup_{s \to 0+} sf(s) = \lim_{s \to 0+} (- s_0) \{ (s s_0^{-1})^{2m+1} - 1 \} \beta(0+) = A\, \alpha(0+).$$

Also

$$s^{-2m} f(s) = s_0^{-2m} f(s_0) - s_0\, s^{-2m-1} \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt.$$

Let us set

$$Q' \equiv \left| s^{-2m-1} \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} \beta(t)\, dt - \right.$$

$$\left. - s^{-2m-1} \{(s\, s_0^{-1})^{2m+1} - 1\} \beta(\infty) \right|,$$

$$\leqslant \left| s^{-2m-1} \int_0^\infty \frac{d}{dt} \left\{ \frac{F[t, s]}{F[t, s_0]} \right\} |\beta(t) - \beta(\infty)|\, dt \right|,$$

$$\leqslant |\, s^{-2m-1} \epsilon' \{(s\, s_0^{-1})^{2m+1} - 1\} \,| + M' \left| s^{-2m-1} \left[\left\{ \frac{F[R, s]}{F[R, s_0]} \right\} - 1 \right] \right|.$$

Therefore

$$\limsup_{s\to\infty} s^{-2m} f(s) = \overline{\lim_{s\to\infty}} \left[s_0^{-2m} f(s_0) - s_0\, s^{-2m-1}\{(s\, s_0^{-1})^{2m+1} - 1\}\, \beta(\infty)\right]$$
$$= 0.$$

Thus the theorem is proved.

THEOREM 4.2b. *If $f(s)$ is defined by convergent integral* (7) *with* $\alpha(0) = 0$, *and*

(1) $\operatorname{Re}(2m + 1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots$

(3) (i) $k - m = \frac{1}{2}$; *or*

(ii) $2m \neq 0$, *or a positive integer*, $k \pm m \neq \frac{1}{2}$; $\operatorname{Re} m < 0$;

then

$$\frac{d^n}{ds^n}\left[s^{-2m} f(s)\right] \sim (-1)^n \frac{\Gamma(2m+n+1)}{\Gamma(m-k+3/2)} \times$$
$$\times\, \alpha(0+)\, s^{-n-1-2m} \quad (s \to 0+;\ n = 1, 2, \ldots) \qquad (25)$$

$$\frac{d^n}{ds^n}\left[s^{-2m} f(s)\right] = o(s^{-n}) \qquad (s \to \infty;\ n = 1, 2, \ldots). \qquad (26)$$

PROOF: From Theorem 4.1b, we have

$$f(s) = \frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \frac{1}{s^2} \int_0^\infty F\left[\begin{matrix} 2m+2,\ 2 \\ m-k+5/2 \end{matrix};\ -\frac{t}{s}\right] \alpha(t)\, dt.$$

But [1, p. 102, (21)]

$$\frac{d}{dz}\left[z^a F(a, b; c; z)\right] = a z^{a-1} F(a+1, b; c; z);$$

and so we have

$$\frac{d}{ds}\left[s^{-2m-2} F\left[\begin{matrix} 2m+2,\ 2 \\ m-k+5/2 \end{matrix};\ -\frac{t}{s}\right]\right]$$
$$= (-1)(2m+2)\, s^{-2m-3} F\left[\begin{matrix} 2m+3,\ 2 \\ m-k+5/2 \end{matrix};\ -\frac{t}{s}\right].$$

Hence, on repeated differentiation, we have

$$\frac{d^n}{ds^n}[s^{-2m}f(s)] = (-1)^n \frac{\Gamma(2m+2+n)}{\Gamma(m-k+5/2)} s^{-2m-2-n} \times$$

$$\times \int_0^\infty F\left[\begin{matrix} 2m+2+n, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] \alpha(t)\, dt.$$

Let us set

$$I' \equiv (-1)^n s^{2m+n+1} \frac{\Gamma(m-k+3/2)}{\Gamma(2m+n+1)} \frac{d^n}{ds^n}[s^{-2m}f(s)] - \alpha(0+),$$

$$= \frac{(2m+n+1)}{s(m-k+3/2)} \int_0^\infty F\left[\begin{matrix} 2m+n+2, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right]\{\alpha(t) - \alpha(0+)\}\, dt,$$

as ([1] p. 79, (4))

$$\int_0^\infty F\left[\begin{matrix} 2m+2+n, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right] dt = \frac{s(m-k+3/2)}{(2m+n+1)}.$$

Given $\epsilon > 0$, we choose δ so small that for $0 \leqslant t \leqslant \delta$

$$|\alpha(t) - \alpha(0+)| < \epsilon,$$

and let

$$M' = \operatorname*{l.u.b.}_{\delta \leqslant t < \infty} |[\alpha(t) - \alpha(0+)]\, t^{-2m-1}|,$$

where M' is finite by Theorem 3.1.

Then following the arguments of Theorem 4.1b we can prove (25).

Similarly (26) can be established.

THEOREM 4.3a. *If $f(s)$ is defined by convergent integral* (8) *and*

(1) Re $(2m+1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots,$ *and*

(3) (i) *$2m$ is zero or a positive integer, or*

(ii) *$2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$, and* Re $m \geqslant 0$, *or*

(iii) $k + m = \frac{1}{2}$,

then

$$f(s) = o(s^{-1}) \quad (s \to 0+), \tag{27}$$

$$f(s) = o(1) \quad (s \to \infty). \tag{28}$$

PROOF. Here let us set

$$\beta(t) = A\, s_0^{-1} \int_{0+}^{t} F[t, s]\, d\,\alpha(t).$$

In the present case (see Theorem 4.1a) $\beta(0+) = 0$, so that

$$f(s) = o(s^{-1}) \quad (s \to 0+);$$

and (28) can also be easily established.

THEOREM 4.3b. *If $f(s)$ is defined by convergent integral* (8) *and*

(1) $\mathrm{Re}\,(2m+1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots$

(3) (i) *$2m$ is positive integer, or*

(ii) *$2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$,* $\mathrm{Re}\, m > 0$, *or*

(iii) $k + m = \frac{1}{2}$;

then

$$f^{(n)}(s) = o(s^{-n-1}) \quad (s \to 0+, n = 1, 2, \ldots) \tag{29}$$

$$f^{(n)}(s) = o(s^{-n}) \quad (s \to \infty, n = 1, 2, \ldots). \tag{30}$$

PROOF. Integrating (8) by parts and differentiating repeatedly, we have

$$f^{(n)}(s) = -A\,(n!)\,\alpha(0+)\,s^{-n-1} + \frac{d^n}{ds^n}\left[\frac{\Gamma(2m+2)}{\Gamma(m-k+5/2)} \times \right.$$

$$\left. \times \frac{1}{s^2}\int_0^\infty F\left[\begin{matrix} 2m+2, 2 \\ m-k+5/2 \end{matrix}; -\frac{t}{s}\right]\alpha(t)\,dt\right]. \tag{31}$$

Using Theorem 4.1b, we have (29) and (30).

THEOREM 4.4a. *If $f(s)$ is defined by convergent integral* (8) *and*

(1) $\mathrm{Re}\,(2m+1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots$

(3) (i) $k - m = \frac{1}{2}$, *or*

(ii) *$2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$,* $\mathrm{Re}\, m < 0$,

then

$$f(s) = o(s^{-1}) \quad (s \to 0+) \tag{33}$$

$$s^{-2m} f(s) = o(1) \qquad (s \to \infty). \tag{34}$$

The proof of this theorem is similar to that of Theorem 4.3a.

THEOREM 4.4b. *If* $f(s)$ *is defined by convergent integral* (8) *and*

(1) $\mathrm{Re}\,(2m+1) > 0$,

(2) $m - k + 3/2 \neq 0, -1, -2, \ldots$

(3) (i) $k - m = \frac{1}{2}$, *or*

(ii) $2m \neq 0$, *or a positive integer*, $k \pm m \neq \frac{1}{2}$; $\mathrm{Re}\, m < 0$;

then

$$\frac{d^n}{ds^n}\,[s^{-2m} f(s)] = o\,(s^{-2m-n-1}) \quad (s \to 0,\, n = 1, 2, \ldots) \tag{35}$$

$$\frac{d^n}{ds^n}\,[s^{-2m} f(s)] = o\,(s^{-n}) \quad (s \to \infty,\, n = 1, 2, \ldots) \tag{36}$$

The proof of this theorem is similar to that of Theorem 4.3b.

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ON LINDELOF'S THEOREMS ON ENTIRE FUNCTIONS

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1. Let $f(z) = z^m e^{Q(z)}. P(z)$, ($m$ integer or zero) be an entire function of finite non-zero order ρ, where $Q(z)$ is a polynomial of degree $\leqslant \rho$ and let

$$P(z) = \prod_{1}^{\infty} E(z/z_n, p),$$

$$E(z/z_n, p) = (1 - z/z_n) \exp\left\{ \frac{z}{z_n} + \frac{z^2}{2z_n} + \dots + \frac{z^p}{pz_n} \right\}$$

the canonical product (c.p.) of genus p. Let $n(r)$ and $M(r)$ have their usual meanings.

$f(z)$ is said to be of maximal (max), mean, or minimal (min) type order ρ according as the $\limsup_{r\to\infty} \dfrac{\log M(r)}{r^\rho}$ is infinite, finite and non-zero, or zero respectively.

In this paper we take $r^\rho L(r) = H(r)$ (say), as a comparison function, where $L(x)$ is a function of x, continuous for all large x, such that $L(ax) \sim L(x)$ for every fixed positive constant a.

It is known that [2] $x^c L(x) \to \infty$, $x^{-c} L(x) \to 0$ for every constant $c > 0$, as $x \to \infty$. Further, $\int_1^r L(t)\,dt \sim r L(r)$; $\int_r^\infty t^{-2} L(t) dt \sim r^{-1} L(r)$.

We call $f(z)$ to be of max 'L-type', mean 'L-type' or min 'L-type' order ρ according as

$$\limsup_{r\to\infty} \frac{\log M(r)}{H(r)} = \tau_L \text{ (say)}$$

is infinite, finite and non-zero or zero respectively.

We shall prove the following analogues of Lindelöf's theorems ([1], 2.9.5, 2.10.1, 2.10.3).

THEOREM 1. *An entire function $f(z)$ of non-integral order ρ is of mean 'L-type' if and only if $n(r) = O(H(r))$ and of min 'L-type' if $n(r) = o(H(r))$.*

Let $S(r) = \sum_{|z_n| \leqslant r} z_n^{-\rho}$ and suppose now that $L(r)$ is non-decreasing.

THEOREM 2. *An entire function $f(z)$ of integral order ρ is of mean 'L-type' if and only if $n(r) = O(H(r))$ and $S(r) = O(L(r))$.*

THEOREM 3. (i) *An entire function $f(z)$ of integral order ρ and genus $p = \rho$ is of min 'L-type' if and only if $n(r) = o(H(r))$ and $\{c_\rho + S(r)/\rho\} = o(L(r))$, where c_ρ is the coefficient of z^ρ in $Q(z)$.*

(ii) *If $p = \rho - 1$, and $c_\rho = 0$ then $f(z)$ will be of min 'L-type'.*

2. We give examples to show that $L(r)$ must necessarily be non-decreasing in Theorems 2 and 3.

EXAMPLE 1. Let ([4], p. 87)

$$F(z) = \prod_2^\infty \left\{1 + \frac{z}{n(\log n)^2}\right\}.$$

Then

$$n(r) \sim r/(\log r)^2; \ \log M(r) \sim r/\log r.$$

Now let $L(r) = (\log r)^{-1}$, and so $F(z)$ is of mean 'L-type', $n(r) = O(H(r))$ but $S(r) \neq O(L(r))$.

EXAMPLE 2. Let ([3], p. 44)

$$f(z) = \prod_2^\infty E(z/a_\nu, p),$$

where

$$a_\nu = -\{\nu(\log \nu)^\alpha\}^{1/\rho}$$

and $\alpha > 1$ if $p = \rho - 1$ and $\alpha \leqslant 1$ if $p = \rho$. Then

$$n(r) \sim \frac{r^\rho}{(\rho \log r)^\alpha}, \quad \log M(r) \sim \frac{1}{\rho^\alpha |1 - \alpha|} \frac{r^\rho}{(\log r)^{\alpha - 1}}.$$

Take now $p = \rho - 1$, $L(r) = (\log r)^{1-\alpha}$; $\alpha > 1$. Then $n(r) = o(H(r))$, $c_\rho \equiv 0$ but $f(z)$ is of mean 'L-type'.

3. We require the following lemmas. Write

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r)}{H(r)} = \frac{\Delta}{\delta} \text{ and } \liminf_{r\to\infty} \frac{\log M(r)}{H(r)} = t_L .$$

Lemma 1. *If $f(z)$ is an entire function of order ρ and type τ_L and lower type t_L, then*

$$\limsup_{r\to\infty} \frac{1}{H(r)} \int_0^r \frac{n(x)}{x}\, dx \leqslant \tau_L \tag{3.1}$$

$$\Delta \leqslant e\,\rho\,\tau_L , \tag{3.2}$$

$$\delta \leqslant \rho\, t_L , \tag{3.3}$$

$$\Delta + \delta \leqslant \Delta \exp(\delta/\Delta) \leqslant e\,\rho\,\tau_L , \tag{3.4}$$

$$\delta < \rho\,\tau_L \text{ if } \delta < \Delta. \tag{3.5}$$

We here prove a little more than what is needed for the proofs of the theorems. The lemma also includes the theorems ([1], 2.5.13, 2.5.16).

Proof. By Jensen's theorem, for every $\epsilon > 0, r > r_0(\epsilon)$

$$\int_0^r \frac{n(x)}{x}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta \leqslant (\tau_L + \epsilon)\, H(r)$$

and (3.1) follows. Further for every $R = ra^{1/\rho}$, $a \geqslant 1$,

$$\left\{\int_0^{r_0} + \int_{r_0}^{r} + \int_r^R\right\} \frac{n(x)}{x}\, dx \leqslant \log M(R)$$

and so for $0 < \delta < \infty$, we get†

$$(\delta - \epsilon)\int_{r_0}^r x^{\rho-1} L(x)\, dx + n(r)\int_r^R \frac{dx}{x} \leqslant K + \log M(R).$$

Hence ([2], Lemma 5),

† In what follows K denotes a positive constant not necessarily the same at each occurrence.

$$\frac{(\delta-\epsilon)\,H(r)}{\rho}+\frac{n(r).\log a}{\rho}\leqslant K+\log M(R)$$

therefore

$$\delta+\Delta\log a\leqslant\rho\,a\,\tau_L \tag{3.6}$$

and

$$\delta\,(1+\log a)\leqslant a\,\rho\,t_L\,. \tag{3.7}$$

The left hand side of (3.6) is maximum when $a=\exp\left\{\frac{\Delta-\delta}{\Delta}\right\}$,

hence

$$\Delta\leqslant\rho\,\tau_L\exp\left(1-\frac{\delta}{\Delta}\right)\leqslant e\,\rho\,\tau_L \tag{3.8}$$

which gives (3.2). Further, taking $a=1$ in (3.7) we get (3.3). Since for $x\geqslant 0, e^x\geqslant 1+x$, we obtain from (3.8)

$$\rho\,\tau_L\geqslant\frac{\Delta}{e}\,e^{\delta/\Delta}\geqslant\frac{\Delta}{e}\left(1+\delta/\Delta\right)$$

and (3.4) follows. Next, $e^x\geqslant ex$ with equality if $x=1$, hence

$$e^{\delta/\Delta}\geqslant e.\,\frac{\delta}{\Delta}\text{ with equality if }\delta=\Delta,$$

and so

$$\tau_L\geqslant\frac{\Delta}{e\,\rho}\,e^{\delta/\Delta}\geqslant\delta/\rho\text{ and }\tau_L>\delta/\rho\text{ if }\delta<\Delta.$$

LEMMA 2. *A canonical product $P(z)$ of order ρ and genus $\rho-1$ is of min 'L-type' if $L(r)$ is non-decreasing.*

PROOF. Since genus $=\rho-1$, the series $\sum_{1}^{\infty} r_n^{-\rho}$ and the integral $\int^{\infty}\frac{n(x)}{x^{\rho+1}}\,dx$ are both convergent. Also since $n(r)$ is non-decreasing,

$$\frac{n(r)}{H(r)}\leqslant\rho\,\frac{n(r)}{L(r)}\int_r^{\infty}\frac{dx}{x^{\rho+1}}<\frac{\rho}{L(r)}\int_r^{\infty}\frac{n(x)}{x^{\rho+1}}\,dx=o(1),$$

as $r\to\infty$. That is $n(r)=o(H(r))$. Further ([4], p. 53), we have

$$\log|P(z)| \leqslant K.r^{\rho}\int_0^{\infty}\frac{n(x)}{x^{\rho}(x+r)}\,dx$$

$$\leqslant K.r^{\rho-1}\int_0^{r}\frac{n(x)}{x^{\rho}}\,dx + K.r^{\rho}\int_r^{\infty}\frac{n(x)}{x^{\rho+1}}\,dx.$$

$$\leqslant K.r^{\rho-1}.\epsilon\int_0^{r}L(x)\,dx + o(H(r))$$

$$\sim K.r^{\rho-1}\epsilon\, rL(r) + o(H(r))$$

and the result follows.

LEMMA 3. *Let* $P_1(z) = \prod_1^N \{1 - z/z_n\}$, *where the points* z_n *lie in the circle* $|z| \leqslant r$. *If* $n(r) = O(H(r))$ *then for* $|z| \leqslant r$

$$\log|P_1(z)| < K.H(r), \quad K = K(\rho).$$

PROOF. The hypothesis implies

$$\int_0^r \frac{n(x)}{x}\,dx \leqslant K\int_0^r x^{\rho-1}L(x)\,dx \sim \frac{K}{\rho}.H(r)$$

and so

$$\sum_1^N \log\frac{r}{r_n} = \int_0^r \frac{n(x)}{x}\,dx \leqslant K.H(r).$$

Now

$$\log|P_1(re^{i\theta})| \leqslant \sum_1^N \log\frac{r}{r_n} + \sum_1^N \log\left(\frac{r_n}{r}+1\right)$$

$$\leqslant K.H(r) + N.\log 2 \leqslant K.H(r)$$

and the result follows for $|z| < r$ by maximum modulus principle.

4. PROOF OF THEOREM 1. The 'only if' part of the theorem follows immediately from Lemma 1, inequality (3.2). So it remains to be proved that $f(z)$ is of mean 'L-type' or min 'L-type' according as $n(r) = O(H(r))$ or $n(r) = o(H(r))$. Since functions of non-integral order are dominated by its c.p. hence we have to deal only with c.p. $P(z)$. Let first $n(r) = O(H(r))$, then for $r \geqslant r_0$ ([4], p. 53)

$$\log|P(z)| \leqslant K.\left\{\int_{r_0}^{r} + \int_{r}^{\infty}\right\} \frac{n(x)\, r^{p+1}}{x^{p+1}(x+r)}\, dx$$

$$\leqslant K.r^p \left\{ \int_{r_0}^{r} x^{-p-1} H(x)\, dx + r \int_{r}^{\infty} x^{-p-2} H(x) dx \right\}$$

and since $p < \rho < p+1$, we have ([2], Lemma 5)

$$\sim K.r^p \left\{ \frac{r^{\rho-p}}{\rho - p} L(r) + r \frac{r^{\rho-p-1}}{p+1-\rho} L(r) \right\}$$

$$\leqslant K.H(r),$$

and the result follows. If $n(r) = o(H(r))$, proceeding similarly as above, we have now, for $r \geqslant r_0$ and with small positive ϵ

$$\log|P(z)| \leqslant K.r^p \left\{ \epsilon \int_{r_0}^{r} x^{-p-1} H(x)\, dx + \epsilon r \int_{r}^{\infty} x^{-p-2} H(x) dx \right\}$$

$$\leqslant \epsilon.H(r)$$

which completes the proof.

PROOF OF THEOREM 2. Denote $u = z/z_n$ and $Q(z) = c_\rho z^\rho + Q_1(z)$ $Q_1(z)$ a polynomial of degree $\leqslant \rho - 1$. If $p = \rho - 1$, then

$$\left|\frac{S(r)}{L(r)}\right| \leqslant \frac{\Sigma\, r_n^{-\rho}}{L(r)} = O(1) \text{ and so } S(r) = O(L(r)).$$

If $f(z)$ is of mean 'L-type' order ρ then (cf. Lemma 1, (3.2)), $n(r) = O(H(r))$. We shall, therefore, discuss the following cases. Consider first $p = \rho$.

(a) If $n(r) = O(H(r))$, $S(r) = O(L(r))$ then $f(z)$ is of mean 'L-type'.

(b) If $f(z)$ is of mean 'L-type', then $n(r) = O(H(r))$ and $S(r) = O(L(r))$.

Let now $p = \rho - 1$.

(c) If $n(r) = O(H(r))$ then $f(z)$ is of mean 'L-type'.

The case (c) follows from Lemma 2. As regards the case (a) we express

$$f(z) = z^m . e^{Q(z)} . \prod_1^N E(u, \rho) \prod_{N+1}^{\infty} E(u, \rho), \tag{4.1}$$

where the integer N will be chosen later on. Hence,

$$\log |f(z)| - \Re \left\{ z^\rho \left(c\rho + \frac{S(r)}{\rho} \right) \right\}$$

$$= \Re \left\{ Q_1(z) + m . \log z \right\} + \sum_1^N \log |E(u, \rho - 1)| + \sum_{N+1}^{\infty} \log |E(u, \rho)|. \tag{4.2}$$

Further ([1], 2.6.6) with $|u| \geqslant 1/k$, $k > 1$ and taking $\rho > 1$,

$$\sum_1^N \log |E(u, \rho - 1)| \leqslant K \sum_1^N |u|^{\rho-1}, \quad K = K(\rho)$$

$$\leqslant K(H(r))^{1-1/\rho} \sum_1^N \frac{1}{n^{1-1/\rho}}$$

since $n < K . H(r_n) < K . r_n^\rho L(r)$.

Next for $\rho = 1$, we have, similarly,

$$\sum_1^N \log |E(u, \rho - 1)| \leqslant \sum_1^N \log (1 + |u|)$$

$$\leqslant \sum_1^N \log \left(1 + \frac{K . H(r)}{n} \right)$$

Let us now choose $N = [H(r)]$, then we have, finally,

$$\sum_1^N \log |E(u, \rho - 1)| \leqslant K . H(r); \; K = K(\rho), \; \rho \geqslant 1 . \tag{4.3}$$

For $|u| < 1/k < 1$, we have ([4], Lemma 6)

$$E(u, \rho) = \exp \left\{ \theta \frac{k}{(k-1)(\rho+1)} |u|^{\rho+1} \right\}, \; |\theta| < 1 .$$

Hence

$$\prod_{N+1}^{\infty} E(u, \rho) = \exp \left\{ \theta \frac{k}{(k-1)^k (\rho+1)} \sum_{N+1}^{\infty} |u|^{\rho+1} \right\} .$$

But

$$\left|\frac{\theta}{\rho+1}\sum_{N+1}^{\infty}|u|^{\rho+1}\right| \leqslant \frac{1}{\rho+1}\sum_{N+1}^{\infty}\left(r/r_n\right)^{\rho+1} = \frac{r^{\rho+1}}{\rho+1}\int_{r_{N+1}}^{\infty}\frac{n(x)}{x^{\rho+2}}\,dx$$

$$\leqslant K.\,r^{\rho+1}\int_{r_{N+1}}^{\infty}\frac{L(x)}{x^2}\,dx$$

$$\sim K.\,r^{\rho+1}\,r_{N+1}^{-1}\,L(r_{N+1})$$

and so

$$\left|\frac{\theta}{\rho+1}\sum_{N+1}^{\infty}|u|^{\rho+1}\right| < K.H(r), \quad K = K(k,\rho). \tag{4.4}$$

Therefore

$$\sum_{N+1}^{\infty}\log|E(u,\rho)| \leqslant K.H(r). \tag{4.5}$$

From (4.4) we also obtain

$$\left|\Re\left\{\frac{\theta}{\rho+1}\sum_{N+1}^{\infty}|u|^{\rho+1}\right\}\right| \leqslant K.H(r).$$

Consequently

$$\sum_{N+1}^{\infty}\log|E(u,\rho)| \geqslant -K.H(r). \tag{4.6}$$

Combining (4.2), (4.3) and (4.5), we have

$$\log|f(z)| - \Re\left\{z^{\rho}\left(c_{\rho}+\frac{S(r)}{\rho}\right)\right\} \leqslant K.H(r),\ \rho \geqslant 1. \tag{4.7}$$

If $S(r) = O(L(r))$, it follows at once from (4.7) that $f(z)$ is of mean 'L-type'.

We now prove the case (b). If $f(z)$ is of mean 'L-type' it easily follows from the inequality (3.2) Lemma 1 that $n(r) = O(H(r))$. To prove the rest, we now express

$$f(z) = z^m.\,e^{Q(z)}.\prod_{1}^{N}(1-u)\prod_{1}^{N}\exp\left(u+\frac{u^2}{2}+\dots+\frac{u^{\rho}}{\rho}\right)\prod_{N+1}^{\infty}E(u,\rho)$$

with $N = [H(r)]$. And so,

$$\Re\left\{z^{\rho}\left(c+\frac{S(r)}{\rho}\right)\right\}+\sum_{1}^{N}\Re\left\{u+\frac{u^2}{2}+\ldots+\frac{u^{\rho-1}}{\rho-1}\right\}+$$

$$+\Re\{Q_1(z)+m\log z\}=\log|f(z)|-\sum_{1}^{N}\log|1-u|-$$

$$-\sum_{N+1}^{\infty}\log|E(u,\rho)|. \qquad (4.8)$$

Further for any given $k>1$ there is always a circle $|z|=r$ in the region $Rk^{-2}\leqslant|z|\leqslant Rk^{-1}$ on which ([4], p. 80)

$$\sum_{1}^{N}\log|1-u|>-K.N, \quad K=K(k) \qquad (4.9)$$

$$>-K.H(r).$$

So, combining (4.6), (4.9) with (4.8) we get,

$$\Re\left\{z^{\rho}\left(c_{\rho}+\frac{S(r)}{\rho}\right)\right\}+\sum_{1}^{N}\Re\left\{u+\frac{u^2}{2}+\ldots+\frac{u^{\rho-1}}{\rho-1}\right\}+O(r^{\rho-1})$$

$$\leqslant\log M(r)+K.H(r)+K.H(r) \quad \text{on } |z|=r,\ Rk^{-2}\leqslant r\leqslant Rk^{-1}$$

Therefore,

$$\Re\left\{z^{\rho}\left(c_{\rho}+\frac{S(r)}{\rho}\right)\right\}+\sum_{1}^{N}\Re\left\{|u|e^{i(\theta-\theta_n)}+\frac{|u|^2}{2}e^{2i(\theta-\theta_n)}+\right.$$

$$\left.+\frac{|u|^{\rho-1}}{\rho-1}e^{i(\rho-1)(\theta-\theta_n)}\right\}\leqslant\{(\tau_L+\epsilon)+K+o(1)\}H(r)$$

for all θ and $r\geqslant r_8(\epsilon)$.

Thus we have

$$\Re\left\{z^{\rho}\left(c_{\rho}+\frac{S(r)}{\rho}\right)\right\}+\sum_{\alpha=1}^{\rho-1}\sum_{1}^{N}\frac{1}{\alpha}(r/r_n)^{\alpha}\cos\alpha(\theta-\theta_n)$$

$$\leqslant\{(\tau_L+\epsilon)+K+o(1)\}H(r), \quad Rk^{-2}\leqslant r\leqslant Rk^{-1}. \qquad (4.10)$$

Further

$$\frac{1}{\alpha}\sum_{1}^{N}\left(\frac{r}{r_n}\right)^{\alpha} = \frac{r^{\alpha}}{\alpha}\int_0^R \frac{dn(x)}{x^{\alpha}}$$

$$= \frac{r^{\alpha}}{\alpha}\left\{\frac{n(R)}{R^{\alpha}} + \alpha\int_0^R \frac{n(x)}{x^{\alpha+1}}\,dx\right\}$$

$$\leqslant \frac{r^{\alpha}}{\alpha}\left\{K.\,R^{\rho-\alpha}L(R) + \alpha K\int_0^R X^{\rho-\alpha-1}L(x)\,dx\right\}$$

$$\sim \frac{K}{\alpha}k^{2(\rho-\alpha)}.\,r^{\rho}L(r) + Kr^{\rho}\frac{R^{\rho-\alpha}}{\rho-\alpha}L(R)$$

$$\sim \frac{K}{\alpha}k^{2(\rho-\alpha)}.\,H(r) + \frac{K}{\rho-\alpha}k^{2(\rho-\alpha)}.\,H(r)$$

and so,

$$\frac{1}{\alpha}\sum_{1}^{N}(r/r_n)^{\alpha} \leqslant \frac{K\rho}{\alpha(\rho-\alpha)}k^{2(\rho-\alpha)}.\,H(r),\ Rk^{-2}\leqslant r\leqslant Rk^{-1}.$$

Consequently,

$$\left|\sum_{\alpha=1}^{\rho-1}\sum_{1}^{N}\frac{1}{\alpha}(r/r_n)^{\alpha}\cos\alpha(\theta-\theta_n)\right| \leqslant \sum_{\alpha=1}^{\rho-1}\sum_{1}^{N}\frac{1}{\alpha}(r/r_n)^{\alpha}.$$

$$\leqslant K\rho H(r)\sum_{\alpha=1}^{\rho-1}\frac{k^{2(\rho-\alpha}}{\alpha(\rho-\alpha)}$$

$$= O(H(r)). \tag{4.11}$$

Hence from (4.10) and (4.11) we have

$$\left|\Re\left\{z^{\rho}\left(c_{\rho}+\frac{S(r)}{\rho}\right)\right\}\right| \leqslant K.\,H(r) \tag{4.12}$$

for all θ and $Rk^{-2}\leqslant r\leqslant Rk^{-1}$, and this implies $S(r) = O(L(r))$ with a fixed k, which completes the proof of the theorem.

Proof of Theorem 3 (i). Again, we shall classify the following two cases :

(a_1) If $n(r) = o(H(r))$ and $\left(c_\rho + \frac{S(r)}{\rho}\right) = o(L(r))$, then $f(z)$ is of min 'L-type'.

(b_1) If $f(z)$ is of min 'L-type' then $n(r) = o(H(r))$ and $\left(c_\rho + \frac{S(r)}{\rho}\right) = o(L(r))$.

The proofs of the cases (a_1) and (b_1) are similar to those of (a) and (b) of Theorem 2, respectively. In fact, here in case (a_1), since now $n(r) < \epsilon . H(r)$, we obtain by moving likewise as in the case (a),

$$\log |f(z)| - \Re \left\{ z^\rho \left(c_\rho + \frac{S(r)}{\rho} \right) \right\} \leqslant \epsilon\, H(r),\ \rho > 1$$

instead of (4.7), while in case (b_1) we obtain, proceeding as in the case (b),

$$\left| \Re \left\{ z^\rho \left(c\rho + \frac{S(r)}{\rho} \right) \right\} \right| \leqslant \epsilon . H(r)$$

instead of (4.12) and we have the results required.

(ii) In view of the fact that $c_\rho = 0$ it is sufficient to show that the c.p. $P(z)$ is of min 'L-type'. Now, if $p = \rho - 1$ then $n(r) = o(H(r))$ and $P(z)$ is of min 'L-type' (cf. Lemma 2). Hence $f(z)$ is at most of min 'L-type'.

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THE GROUPS OF WORD PRESERVING PERMUTATIONS OF A GROUP†

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1. Introduction. Throughout this paper let G be a group and let F be a free group of countable rank with a set I of independent generators. Thus

$$F = \left\{ \prod_{i=1}^{n} X_i^{e_i} : X_i \in I,\, e_i = \pm 1 \text{ and } n \text{ is a positive integer} \right\}.$$

Let $\mathfrak{P}$ be the group of all permutations of the set G. For each $\prod_{i=1}^{n} X_i^{e_i}$ in F, define $[\Pi X_i^{e_i}] = \{\alpha \in \mathfrak{P} : (\Pi\, y_i^{e_i})\, \alpha = \Pi\, (y_i\, \alpha)^{e_i}$ for all subsets $\{y_i\}_{i=1}^{n}$ of G for which $y_i = y_j$ if $X_i = X_j\}$. Each $[\Pi X_i^{e_i}]$ is a subgroup of $\mathfrak{P}$ that contains the group of all automorphisms of G. The notation $[\Pi X_i^{e_i}]$ does not imply that the X_i are distinct. However, elements of I that are denoted by different letters will be assumed to be distinct—for example, $[XY]$, $[X^{-1}Y]$ and $[XY^{-1}Z]$. In particular, $\mathfrak{P} = [X]$, $[XY]$ is the group of all automorphisms of G, $[XY^{-1}Z]$ is the holomorph of G, and $[XYX] \cap [XX^{-1}]$ is the group of all semi-automorphisms of G ([1] p. 479).

In this paper the structure of the groups $[\Pi X_i^{e_i}]$ and the relationships between these groups and G are investigated. Any restriction on G places restrictions on the $[\Pi X_i^{e_i}]$. For example, Dinkiness [1] has shown that if G is a symmetric or alternating group, then $[XYX] \cap [XX^{-1}]$ consists of the automorphisms and anti-automorphisms of G. In section 4 the structure of $[X^n]$ is determined for all groups G that satisfy: $a^n = b^n$ implies $a = b$ for all a, b in G. Conversely, information about the $[\Pi X_i^{e_i}]$ leads to information about G. For example, Mills [5] has shown that if

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G and G' are finitely generated abelian groups and their holomorphs are isomorphic, then G is isomorphic to G'. In section 3 it is shown that $[XX^{-1}] \subset [X^{n+1}]$ if and only if $g^n = e$ for all g in G or $g^{n+1} = e$ for all g in G or G is of order 3. Section 5 deals with the holomorph of G and the groups $[\Pi\, X_i^{e_i}]$ that contain the holomorph. Also, the structures of all groups of the form $[X^{e_1} Y^{e_2} Z^{e_3}]$, where $e_i = \pm 1$, are determined.

NOTATION. Let e and θ be the identity elements of G and $[X]$ respectively. θ will also be used to denote the identity of other transformation groups, and it will be clear from the context to which group θ belongs. π will always denote the anti-automorphism of G that maps each element onto its inverse, $g\pi = g^{-1}$ for all g in G. If A is a group, then $Z(A)$ will denote the center of A. For any set S, $|S|$ will denote the cardinality of S. $\square$ will denote the null set.

2. Some basic lemmas.

$\prod_{i=1}^{m} X_i^{e_i}$ and $\prod_{i=1}^{n} Y_i^{d_i}$ will always denote elements in F, where the X_i and Y_i belong to I and the e_i and d_i equal ± 1. $\prod_{i=1}^{m} X_i^{e_i}$ will be replaced by $\Pi\, X_i^{e_i}$ whenever this causes no confusion.

THEOREM 2.1. *For each* $\Pi\, X_i^{e_i}$ *in* F, $[\Pi\, X_i^{e_i}]$ *is a subgroup of* $[X]$ *that contains* $[XY]$.

This follows immediately from the definition of $[\Pi\, X_i^{e_i}]$.

LEMMA 2.1. *If* $\prod_{i=1}^{m} X_i^{e_i} = \prod_{i=1}^{n} Y_i^{d_i}$ *in* F, *then* $[\Pi\, X_i^{e_i}] = [\Pi\, Y_i^{d_i}]$ *in* $[X]$.

PROOF. It suffices to prove the lemma for the special case where $\Pi\, X_i^{e_i}$ is a simple reduction of $\Pi\, Y_i^{d_i}$, since $\Pi\, X_i^{e_i}$ can be obtained from $\Pi\, Y_i^{d_i}$ by a finite sequence of simple reductions and expansions. If $\Pi\, X_i^{e_i}$ is a simple reduction of $\Pi\, Y_i^{e_i}$, then $n = m + 2$ and

$$\Pi\, Y_i^{d_i} \equiv X_1^{e_1} \dots X_i^{e_i}\, Y\, Y^{-1}\, X_{i+1}^{e_{i+1}} \dots X_m^{e_m},$$

where Y or Y^{-1} belongs to I. Pick $z, z_1, z_2, \ldots, z_m$ in G such that $z_i = z_j$ if $X_i = X_j$. If $\alpha \in [\Pi X_i^{e_i}]$, then

$$(z_1^{e_1} \ldots z_i^{e_i} z z^{-1} z_{i+1}^{e_{i+1}} \ldots z_m^{e_m}) \alpha = \left(\prod_{i=1}^{m} z_i^{e_i} \right) \alpha = \prod_{i=1}^{m} (z_i \alpha)^{e_i}$$

$$= (z_1 \alpha)^{e_1} \ldots (z_i \alpha)^{e_i} (z \alpha) (z \alpha)^{-1} (z_{i+1} \alpha)^{e_{i+1}} \ldots (z_m \alpha)^{e_m}.$$

Thus $\alpha \in [\Pi Y_i^{d_i}]$. Conversely if $\alpha \in [\Pi Y_i^{d_i}]$, then

$$\left(\prod_{i=1}^{m} z_i^{e_i} \right) \alpha = (z_1^{e_1} \ldots z_i^{e_i} z z^{-1} z_{i+1}^{e_{i+1}} \ldots z_m^{e_m}) \alpha$$

$$= (z_1 \alpha)^{e_1} \ldots (z_i \alpha)^{e_i} (z \alpha) (z \alpha)^{-1} (z_{i+1} \alpha)^{e_{i+1}} \ldots (z_m \alpha)^{e_m}$$

$$= \prod_{i=1}^{n} (z_i \alpha)^{e_i}.$$

Therefore $\alpha \in [\Pi X_i^{e_i}]$.

Therefore we have a mapping $\mu = \mu(I)$ of F into the set of all subgroups of $[X]$ that contain $[XY]$. Since $[X] = [Y]$, μ is not 1-1. We investigate the structures of the groups in $F\mu$ and attempt to recover G in $F\mu$. F and the set I of generators will be fixed. Thus $[f]$, for f in F, is a well-defined subgroup of $[X]$. If $[f]$ is to be investigated for two groups G and G', then the corresponding groups will be denoted by $[f]_G$ and $[f]_{G'}$ respectively.

Lemma 2.2. $[XX^{-1}] = [X^{-1}X] = \{\alpha \in [X] : e\alpha = e\}$. $[XX^{-1}]$ *will be denoted by* $[X^0]$.

Proof. $[XX^{-1}] = [X^{-1}X]$ by Lemma 2.1. If $\alpha \in [XX^{-1}]$, then $e\alpha = (ee^{-1})\alpha = e\alpha(e\alpha)^{-1} = e$. If $\alpha \in [X]$ and $e\alpha = e$, then $(xx^{-1})\alpha = e\alpha = e = x\alpha(x\alpha)^{-1}$ for all x in G.

For each f in F, $f = \prod_{i=1}^{n} X_i^{e_i}$, and $\sum_{i=1}^{n} e_i$ is uniquely determined by f and I.

Lemma 2.3. *Let* $\alpha \in [f]$, *where* $f = \Pi X_i^{e_i}$ *and* $\Sigma e_i = k$.

(a) $[f] \subseteq [X^k]$ *for all* G.

(b) $(e\alpha)^{k-1} = e$ *for all* G.

(c) $[f] \subseteq [X^0]$ *for all* G *if and only if* $k = 0$ *or* 2.

(d) *If no element in* G *has order that divides* $k-1$, *then* $[f] \subseteq [X^0]$.

(e) *If* $[f] \subseteq [X^0]$, *then no element in the center of* G *has order that divides* $k-1$.

(f) *If* G *is abelian, then* $[f] \subseteq [X^0]$ *if and only if no element in* G *has order that divides* $k-1$.

PROOF. (a) $x^k\alpha = (x^{e_1}x^{e_1}\dots x^{e_n})\alpha = (x\alpha)^{e_1}\dots(x\alpha)^{e_n} = (x\alpha)^k$ for all x in G. (b) $e\alpha = e^k\alpha = (e\alpha)^k$. (c) If $k = 0$ or 2, then by (b), $e\alpha = e$. Hence $\alpha \in [X^0]$ for all G. Conversely suppose that $[f] \subseteq [X^0]$ for all G. Assume (by way of contradiction) that $k \neq 0$ or 2. Pick a group G with an element $g \neq e$ in $Z(G)$ for which $g^{k-1} = e$. Define $x\beta = gx$ for all x in G. Pick $\{x_i\}_{i=1}^n \subseteq G$ so that $x_i = x_j$ if $X_i = X_j$. Then $\Pi\,(x_i\beta)^{e_i} = \Pi\,(gx_i)^{e_i} = (\Pi\, x_i^{e_i})g^k = (\Pi x_i^{e_i})\beta$. Also, $e\beta = g \neq e$. Therefore $\beta \in [f] \setminus [X^0]$, a contradication. (d) is an immediate consequence of (b). In order to prove (e), suppose that $g \in Z(G)$ and $g^{k-1} = e$. Then define β as in the proof of (c). $\beta \in [f] \subseteq [X^0]$. Thus $e = e\beta = g$. Finally, (f) is an immediate consequence of (d) and (e).

THEOREM 2.2. *The following statements are equivalent.*

(i) G *is abelian.*

(ii) $[X_1^{e_1}\dots X_n^{e_n}] = [X_{\alpha_1}^{e_{\alpha_1}},\dots X_{\alpha_n}^{e_{\alpha_n}}]$ *for all choices of* n, X_i *and* e_i, *and all permutations* $\alpha_1,\dots,\alpha_n$ *of* $1,\dots,n$.

(iii) $[XYX^{-1}Y^{-1}] = [X^0]$.

PROOF. Suppose that G is abelian. Let $\prod_{i=1}^{n} X_i^{e_i} \in F$, and let $\alpha_1,\dots,\alpha_n$ be a permutation of $1,\dots,n$. Let $\beta \in [\Pi\, X_i^{e_i}]$ and pick $\{y_i\}_{i=1}^n \subseteq G$ so that $y_i = y_j$ if $X_i = X_j$. Then $(\Pi\, y_{\alpha_i}^{e_{\alpha_i}})\beta = (\Pi\, y_i^{e_i})\beta = \Pi(y_i\beta)^{e_i} = \Pi\,(y_{\alpha_i}\beta)^{e_{\alpha_i}}$. Thus $[\Pi\, X_i^{e_i}] \subseteq [\Pi\, X_{\alpha_i}^{e_{\alpha_i}}]$. Similarly $[\Pi\, X_{\alpha_i}^{e_{\alpha_i}}] \subseteq [\Pi\, X_i^{e_i}]$. Thus (i) implies (ii). If (ii) is true, then $[XYX^{-1}Y^{-1}] = [XX^{-1}YY^{-1}]$, and by Lemma 2.1, $[XX^{-1}YY^{-1}] = [X^0]$. Finally suppose that $[XYX^{-1}Y^{-1}] = [X^0]$. If $x^2 = e$ for all x in G, then G is abelian. Otherwise let g in G have order greater than 2, and consider x, y in $G \setminus \{e\}$ such that

$x \neq y$. Then there exists an α in $[X^0]$ such that $x\alpha = g$ and $y\alpha = g^2$. $(xyx^{-1}y^{-1})\alpha = x\alpha y\alpha(x\alpha)^{-1}(y\alpha)^{-1} = e = e\alpha$. Thus $xyx^{-1}y^{-1} = e$ and G is abelian.

Note that by property (a) of Lemma 2.3, (iii) can be replaced by $[XYX^{-1}Y^{-1}] \supseteq [X^0]$. If G is abelian, then F can be replaced by a free abelian group.

An element f in F is called *symmetric* (with respect to I) if $f = \prod_{i=1}^{n} X_i^{e_i} = \prod_{j=0}^{n-1} X_{n-j}^{e_{n-j}}$. If $\prod_{j=0}^{n-1} X_{n-j}^{e_{n-j}} = \prod_{i=1}^{n} X_i^{e_i} = \prod_{i=1}^{m} Y_i^{d_i}$, then $\prod_{i=1}^{m} Y_i^{d_i} = \prod_{j=0}^{m-1} Y_{m-j}^{e_{m-j}}$. This is clear if $\Pi X_i^{e_i}$ is a simple reduction or expansion of $\Pi Y_i^{d_i}$, and $\Pi X_i^{e_i}$ can be obtained from $\Pi Y_i^{d_i}$ by a finite sequence of simple reductions or expansions. Thus the concept of symmetry is independent of the particular representation of f. Moreover if $\Pi X_i^{e_i}$ is symmetric and completely reduced, then $\Pi X_{n-j}^{e_{n-j}}$ is identically equal to $\Pi X_i^{e_i}$.

THEOREM 2.3. *For f in F the following statements are equivalent.* (a) *f is symmetric.* (b) *$[f]$ contains all the anti-automorphisms of G for every G.* (c) *$[f]$ contains all the anti-automorphisms of G when $G = F$.* (d) *π belongs to $[f]$ for every G.* (e) *π belongs to $[f]$ when $G = F$.*

PROOF. First assume that f is symmetric. If $f = XX^{-1}$, then by Lemma 2.2, (b) is satisfied. If $f \neq XX^{-1}$, then $f = \Pi X_i^{e_i} \equiv \Pi X_{n-j}^{e_{n-j}}$, where $\Pi X_i^{e_i}$ is reduced. Select $\{y_i\}_{i=1}^{n} \subseteq G$ such that $y_i = y_j$ if $X_i = X_j$, and let α be an anti-automorphism of G. $(\Pi y_i^{e_i})\alpha = \Pi(y_{n-j}\alpha)^{e_{n-j}} \equiv \Pi(y_i\alpha)^{e_i}$. Thus (a) implies (b). Clearly (b) implies (c), (c) implies (e), (b) implies (d) and (d) implies (e). Thus it suffices to prove that (e) implies (a). Suppose that $G = F$, and $\pi \in [f]$. $f = \Pi X_i^{e_i}$. $(\Pi X_i^{e_i})^{-1} = (\Pi X_i^{e_i})\pi = \Pi(X_i\pi)^{e_i} = \Pi X_i^{-e_i}$. Thus $\Pi X_i^{e_i} = (\Pi X_i^{-e_i})^{-1} = \Pi X_{n-j}^{e_{n-j}}$.

COROLLARY. *$[X^{-1}]$ is the centralizer of π in $[X]$. Thus if f is symmetric, then $[f] \subseteq [X^{-1}]$ if and only if π belongs to $Z([f])$.*

PROOF. Let $g \in G$ and $\alpha \in [X]$. If $\pi\alpha = \alpha\pi$, then $g^{-1}\alpha = g\pi\alpha = g\alpha\pi = (g\alpha)^{-1}$; hence $\alpha \in [X^{-1}]$. If $\alpha \in [X^{-1}]$, then $g\pi\alpha = g^{-1}\alpha = (g\alpha)^{-1} = g\alpha\pi$.

It will be shown in Lemma 4.3 that $\{\theta, \pi\}$ is the center of $[X^{-1}]$ provided that $|\{g \in G : g^2 = e\}| \neq 2$.

LEMMA 2.4. *If* $\left[\prod_{i=1}^{n} X_i^{e_i}\right] \subseteq [X^{-1}]$, *then* $\left[\prod_{i=1}^{n} X_i^{e_i}\right] \subseteq \left[\prod_{i=1}^{n} X_i^{d_i}\right]$, *where* $d_i e_i = d_j e_j$ *if* $X_i = X_j$.

PROOF. The restriction on the d_i simply means that if $X_i = X_j$, then either $d_i = e_i$ and $d_j = e_j$ or $d_i = -e_i$ and $d_j = -e_j$. Suppose that $Y = X_{\alpha_1} = X_{\alpha_2} = \ldots = X_{\alpha_k}$ and the remaining X_i do not equal Y. Let $\Pi X_i^{d_i}$ be the element in F obtained from $\Pi X_i^{e_i}$ by replacing each of the $X_{\alpha_i}{}^{e_{\alpha_i}}$ by $X_{\alpha_i}{}^{-e_{\alpha_i}}$. Then $[\Pi X_i^{e_i}] \subseteq [\Pi X_i^{d_i}]$. For, pick $\{y_i\}_{i=1}^{n} \subseteq G$ such that $y_i = y_j$ if $X_i = X_j$, and pick $\beta \in [\Pi X_i^{e_i}]$. Then $\Pi y_i^{d_i} = \Pi z_i^{e_i}$, where $z_{\alpha_i} = y_{\alpha_i}{}^{-1}$ for $i = 1, 2, \ldots, k$ and $z_j = y_j$ otherwise. Thus $(\Pi y_i^{d_i})\beta = (\Pi z_i^{e_i})\beta = \Pi (z_i \beta)^{e_i}$. But $(z_{\alpha_i} \beta)^{e_i} = (y_{\alpha_i}{}^{-1} \beta)^{e_i} = (y_{\alpha_i} \beta)^{-e_{\alpha_i}} = (y_{\alpha_i} \beta)^{d_{\alpha_i}}$. Therefore $\Pi (z_i \beta)^{e_i} = \Pi (y_i \beta)^{d_i}$. By repeating this process at most n times, the lemma is proven.

LEMMA 2.5. $\left[\prod_{i=1}^{n} X_i^{e_i}\right] \subseteq \left[\prod_{i=1}^{n} Y_i^{e_i}\right]$, *where* $Y_i = Y_j$ *if* $X_i = X_j$.

PROOF. Pick $\alpha \in [\Pi X_i^{e_i}]$ and $\{y_i\}_{i=1}^{n} \subseteq G$ so that $y_i = y_j$ if $Y_i = Y_j$. Then $y_i = y_j$ if $X_i = X_j$, and $(\Pi y_i^{e_i})\,\alpha = \Pi (y_i\, \alpha)^{e_i}$. Therefore $\alpha \in [\Pi Y_i^{e_i}]$. For example, $[XYZ] \subseteq [XYX] \subseteq [X^3]$.

LEMMA 2.6. (a) $[X^m] \cap [X^n] \subseteq [X^{mn}]$ *for all integers* m *and* n. (b) *If* $n_1, n_2, \ldots, n_k$ *are integers and* q *is a product of a subsequence of the* n_i, *then* $\bigcap_{i=1}^{k} [X^{n_i}] \subseteq [X^q]$. (c) $[X^m] \subseteq [X^{m^n}]$ *for all integers* m *and* $n \geqslant 0$.

PROOF. Let $\alpha \in [X^m] \cap [X^n]$ and let $x \in G$. $(x^{mn})\,\alpha = (x^m)^n\,\alpha = (x^m\,\alpha)^n = (x\,\alpha)^{mn}$. Thus (a) is true and (b) is an immediate consequence of (a). (c) is obvious when $n = 0$, and if $n > 0$, then (c) follows from (b) by letting $n = k$ and $m = n_1 = n_2 = \ldots = n_k$.

Consider $\prod_{i=1}^{n} X_i^{e_i}$ in F. This is composed of distinct elements $Z_1, Z_2, \ldots, Z_m$ of I, where $m \leqslant n$. $\sum_{i=1}^{n} e_i = \sum_{i=1}^{m} a_i$, where a_i is the sum of the exponents of Z_i.

LEMMA 2.7. (a) $[\Pi X_i^{e_i}] \cap [X^0] \subseteq \bigcap_{i=1}^{n} [X^{a_i}]$. (b) *If a subsequence of the* a_i *has sum or product equal to* q, *then* $[\Pi X_i^{e_i}] \cap [X^0] \subseteq [X^q]$. (c) $[\Pi X_i^{e_i}] \cap [X^0] \subseteq [\Pi Y_i^{d_i}]$, *where* $\Pi Y_i^{d_i}$ *is an element obtained from* $\Pi X_i^{e_i}$ *by removing completely one or more of the* Z_i. *For example,* $[XY^3X^{-1}Z^2X] \cap [X^0] \subseteq [XY^3X^{-1}X] = [XY^3]$.

PROOF. Let $\alpha \in [\Pi X_i^{e_i}] \cap [X^0]$. Replace all of the X_i except Z_j in $\Pi X_i^{e_i}$ by e, and replace the Z_j by some element y in G. Then clearly $y^{a_j}\alpha = (y\alpha)^{a_j}$. Thus (a) is true, and if q is a product of a subsequence of the a_i, then $[\Pi X_i^{e_i}] \cap [X^0] \subseteq [X^q]$ by property (b) of Lemma 2.6. If $q = a_{\alpha_1} + \ldots + a_{\alpha_\kappa}$, where $a_{\alpha_1}, \ldots, a_{\alpha_\kappa}$ is a subsequence of the a_i, then replace the Z_{α_i} by y and replace all the other X_i by e. Then clearly $y^q\alpha = (y\alpha)^q$. Therefore (b) is true. Finally suppose that $\prod_{i=1}^{h} Y_i^{d_i}$ is the result of removing Z_r from $\Pi X_i^{e_i}$. Choose $\{y_i\}_{i=1}^{h} \subseteq G$ such that $y_i = y_j$ if $Y_i = Y_j$. For $i = 1, \ldots, n$ define x_i: $x_i = e$ if $X_i = Z_r$ and $x_i = y_j$ if $X_i \neq Z_r$ and $X_i = Y_j$. Then $x_i = x_j$ if $X_i = X_j$, and $\left(\prod_{i=1}^{h} y_i^{d_i}\right)\alpha = \left(\prod_{i=1}^{n} x_j^{e_i}\right)\alpha = \prod_{i=1}^{n} (x_i\alpha)^{e_i} = \prod_{i=1}^{h} (y_i\alpha)^{d_i}$. Therefore $[\Pi X_i^{e_i}] \subseteq [\Pi Y_i^{d_i}]$.

3. Some properties of G **that are equivalent to relationships between subgroups of** $[X]$ **of the form** $[X^n]$. Let n be a fixed but arbitrary integer.

LEMMA 3.1. *If* $g^n = e$ *for all* g *in* G, *then* $[X^p] = [X^q]$ *for all integers* p *and* q *such that* $p - q = n$.

PROOF. For x in G, $x^p = x^q$. If $\alpha \in [X^p]$, then $x^q\alpha = x^p\alpha = (x\alpha)^p = (x\alpha)^q$, and if $\alpha \in [X^q]$, then $x^p\alpha = x^q\alpha = (x\alpha)^q = (x\alpha)^p$.

In proving the theorems in this section we make use of the following notation: $\alpha(ab)$, $\alpha(abc)$, and $\alpha(ab)\,(dc)$ denote permutations of G that permute a, b, c and d as indicated and leave the remaining elements of G fixed.

THEOREM 3.1. $[X^n] = [X]$ *if and only if* $x^{n-1} = e$ *for all* x *in* G.

PROOF. Suppose that $[X^n] = [X]$. For x in G, $\alpha = \alpha(e\,x) \in [X]$. $x = e\alpha = e^n\alpha = (e\alpha)^n = x^n$. Therefore $x^{n-1} = e$. The converse follows from Lemma 3.1.

COROLLARY. *If* $[X] = [X^n]$, *then* $[X^{m+1}] = [X^{m+n}]$ *for all integers* m.

The converse of this corollary is false. For if $|G| = 3$, then $[X^3] = [X^0] = [X^2] = [XY]$ and $[X] \neq [X^3]$. G is of bounded order if and only if $[X^{m+1}] = [X]$ for some integer $m \geqslant 1$. The smallest integer $n(G) \geqslant 1$ such that $[X^{n(G)+1}] = [X]$ is the least common multiple of the orders of the elements in G. If G is an abelian group of bounded order, then there exists a g in G with order $n(G)$. In particular, G is a finite cyclic group of order n if and only if $[XYX^{-1}\,Y^{-1}] = [X^0]$, there exists an integer $m \geqslant 1$ such that $[X^{m+1}] = [X]$, $n(G)\,! = |\,[X]\,|$ and $n = n(G)$.

THEOREM 3.2. *The following statements are equivalent.*

(a) G *contains no elements of order* $2, 3, \ldots, n$.

(b) $[X^{-k}] \subseteq [X^0]$ *for* $k = 1, 2, \ldots, n-1$.

(c) $[X^{1-p}] \subseteq [X^0]$ *for all positive primes* $p \leqslant n$.

PROOF. Let $\beta \in [X^{-k}]$. By (b) of Lemma 2.3, $(e\beta)^{k+1} = e$. Thus (a) implies (b), and clearly (b) implies (c). Finally suppose that (c) is satisfied, and assume (by way of contradiction) that there exists an $x \neq e$ in G with order $\leqslant n$. Then without loss of generality x has prime order p and there are no elements in G with order less than p. $\alpha = \alpha\,(ex) \notin [X^0]$, but we show that $g^{1-p}\,\alpha = (g\alpha)^{1-p}$ for all g in G, a contradition. $e^{1-p}\alpha = e\alpha = x = x^{1-p} = (e\alpha)^{1-p}$. $x^{1-p}\,\alpha = x\alpha = e = e^{1-p} = (x\alpha)^{1-p}$. If g in G has infinite order, then $g^{1-p}\alpha$

$= g^{1-p} = (g\alpha)^{1-p}$. Finally assume that $g \neq e$ or x has finite order m. Since $m \geqslant p$, $g^{1-p} \neq e$. If $g^{1-p} = x$, then $g^{p(p-1)} = x^{-p} = e$, and m divides $p(p-1)$. $p(p-1) = pq_1^{n_1} \dots q_t^{n_t}$, where the q_i are primes less than p. Thus either $m = p$ or one of the q_i divides m. If $m = p$, then $x = g^{1-p} = g$. If q_i divides m, then G contains an element of order q_i. Thus if $g \neq e$ or x, then $g^{1-p} \neq e$ or x. Hence $g^{1-p}\alpha = g^{1-p} = (g\alpha)^{1-p}$.

COROLLARY. *G is torsion free if and only if $[X^{1-p}] \subseteq [X^0]$ for all primes p.*

If x in G has finite order m, then $(m, p) = 1$ for all primes $p \leqslant n$ provided that (a) is satisfied. If $|G| = m$, then m is odd if and only if $[X^{-1}] \subseteq [X^0]$.

THEOREM 3.3. *$[X^n] = [X^0]$ if and only if $x^n = e$ for all x in G or $|G| = 3$ and 3 does not divide $n - 1$.*

PROOF. If $x^n = e$ for all x in G, then by Lemma 3.1, $[X^n] = [X^0]$. Suppose that $|G| = 3$ and $3 \nmid n-1$. $[X^0] = [XY] \subseteq [X^n]$, since $|[XY]| = |[X^0]| = 2$. If $\alpha \in [X^n]$, then $(e\alpha)^{n-1} = e$. Then $e\alpha = e$, and $[X^n] \subseteq [X^0]$. Conversely suppose that $[X^n] = [X^0]$. If $n = 0$, then $x^n = e$ for all x in G. If $n = 1$, then by Theorem 3.1, $|G| = 1$. If $n = -1$, then by Theorem 3.2, G contains no elements of order 2. If g in G has order greater than 3, then $\alpha = \alpha\ (gg^2) \in [X^0] = [X^{-1}]$, but $(g\alpha)^{-1} = g^{-2} \neq g^{-1} = g^{-1}\alpha$. Thus $|G| = 1$ or G consists of elements of order 3. If $|G| > 3$, then there exist elements x, y in G of order 3 such that $\{x, x^2\} \cap \{y, y^2\} = \square$. $\beta = \beta\,(xy)$ is in $[X^0]$, but $(x\beta)^{-1} = y^{-1} = y^2 \neq x^2 = x^2\beta = x^{-1}\beta$. Thus $|G| = 1$ or 3 and $3 \nmid -2 = n - 1$.

Suppose that $|n| > 1$, and let $x \neq e$ be an element in G. *Then x has finite order that divides $n-1$ or n or $n+1$.* For assume that this is false. Then $x^n \neq x$ or x^{-1} and $x^2 \neq e$. Let $\alpha = \alpha\,(xx^{-1})$ and $\beta = \beta(xx^2)$. Then $\alpha, \beta \in [X^0]$. $x^n = x^n\alpha = (x\alpha)^n = x^{-n}$. Thus $e = e\beta = x^{2n}\beta = (x^2)^n\beta = (x^2\beta)^n = x^n$, a contradiction. *$x^k = e$ for all x in G, where k is fixed and equal to $n-1$ or n or $n+1$.* For let x, y in G $(x \neq e \neq y)$ have orders $\bar{x}$ and $\bar{y}$ respectively. Then $\gamma = \gamma\,(xy) \in [X^0]$.

If $\bar{x} \mid n$, then $e = e\gamma = x^n\gamma = (x\gamma)^n = y^n$. If $\bar{x} \mid n - 1$, then $y = x\gamma = x^n\gamma = (x\gamma)^n = y^n$. *If $x^{n-1} = e$ for all x in G, then $|G| = 1$.* For by Lemma 3.1, $[X] = [X^n] = [X^0]$, and by Theorem 3.1, $x^{-1} = e$ for all x in G. If $x^{n+1} = e$ for all x in G, then by Lemma 3.1, $[X^{-1}] = [X^n] = [X^0]$. Hence by the above argument for $n = -1$, $|G| = 1$ or 3. If $|G| = 3$, then $3 \mid n + 1$, hence $3 \nmid n - 1$.

Corollary. *G is of bounded order if and only if $[X^0] \subseteq [X^k]$ for some integer $k \neq 0$ or 1.*

Proof. If G is of bounded order, then there exists an integer $k > 1$ such that $x^k = e$ for all x in G. Hence $[X^k] = [X^0]$. Conversely suppose that $[X^0] \subseteq [X^k]$ for some $k \neq 0$ or 1. Suppose that g in G has order greater than $|k| + 2$. $\alpha = \alpha(gg^2g^k) \in [X^0] \subseteq [X^k]$. $g = g^k\alpha = (g\alpha)^k = g^{2k}$. Therefore $g^{2k-1} = e$, and G is bounded.

Theorem 3.4. $[X^0] \subseteq [X^{n+1}]$ *if and only if G satisfies one of the following.* (a) $g^n = e$ *for all g in G.* (b) $g^{n+1} = e$ *for all g in G.* (c) $|G| = 3$.

Proof. If G satisfies (a), then by Lemma 3.1, $[X^0] \subseteq [X] = [X^{n+1}]$. If G satisfies (b), then by Theorem 3.3, $[X^0] = [X^{n+1}]$. If $|G| = 3$, then $[X^0] = [XY] \subseteq [X^{n+1}]$.

Conversely suppose that $[X^0] \subseteq [X^{n+1}]$. If $n = 0$, then G satisfies (a), and if $n = -1$, then G satisfies (b). Next assume that $n \neq 0$ or -1. If $e \neq g$ in G has order that divides n, then G satisfies (a). For $(g\alpha)^{n+1} = g^{n+1}\alpha = g\alpha$ for all α in $[X^0]$. If $e \neq g$ in G has order that divides $n + 1$, then G satisfies (b), since $e = e\alpha = g^{n+1}\alpha = (g\alpha)^{n+1}$ for all α in $[X^0]$. Suppose (by way of contradiction) that $|G| \neq 3$ and that there exists an x in G with order that does not divide n or $n + 1$. Then $x^{n+1} \neq x$ or e and there exists a y in G such that $y \neq x^{n+1}$, x or e. $\alpha = \alpha(x^{n+1}y) \in [X^0]$, but $x^{n+1}\alpha = y \neq x^{n+1} = (x\alpha)^{n+1}$. Thus $\alpha \notin [X^{n+1}]$, a contradiction.

Lemma 3.2. *Let S be a subset of G that is closed with respect to the inverse operation, and let α be a permutation of S such that $(s\alpha)^{-1} = s^{-1}\alpha$ for all s in S. Then $\alpha^* \in [X^{-1}]$, where $x\alpha^* = x\alpha$ if $x \in S$ and $x\alpha^* = x$ if $x \in G \sim S$.*

Proof. Clearly $\alpha^* \in [X]$. If $x \in S$, then $x^{-1} \in S$, and $x^{-1}\alpha^* = x^{-1}\alpha = (x\,\alpha)^{-1} = (x\,\alpha^*)^{-1}$. If $x \in G \smallsetminus S$, then $x^{-1} \in G \smallsetminus S$, and $x^{-1}\alpha^* = x^{-1} = (x\,\alpha^*)^{-1}$.

Theorem 3.5. $[X^n] = [X^{-1}]$ *if and only if* $x^{n+1} = e$ *for all* x *in* G *or* $|G| = 3$ *and* 3 *does not divide* $n - 1$.

Proof. If $x^{n+1} = e$ for all x in G, then by Lemma 3.1, $[X^n] = [X^{-1}]$. Suppose that $|G| = 3$ and $3 \nmid n - 1$. Then by Theorem 3.3, $[X^n] = [X^0]$, and by Theorem 3.2, $[X^{-1}] \subseteq [X^0]$. But in this case $[X^0] = [XY]$. Therefore $[X^{-1}] = [X^0] = [X^n]$.

Conversely suppose that $[X^n] = [X^{-1}]$. If $n = 1$, then by Theorem 3.1, $x^2 = e$ for all x in G. If $n = 0$, then by Theorem 3.3, $|G| = 1$ or $|G| = 3$ and $3 \nmid -1$. If $n = -1$, then $g^{-1+1} = e$ for all g in G. Suppose that $|n| > 1$, and consider $e \neq x$ in G. *Then* x *has finite order that divides* $n - 1$ *or* n *or* $n + 1$. For assume that this is false. Then $x^n \neq e$, x or x^{-1}. $\alpha = \alpha(xx^{-1}) \in [X^{-1}]$. Thus $x^n = x^n\,\alpha = (x\,\alpha)^n = x^{-n}$. Hence $x^{2n} = e$, and $\beta = \beta(e\,x^n) \in [X^{-1}]$. $e = x^n\beta = (x\,\beta)^n = x^n$, a contradiction. $x^k = e$ *for all* x *in* G, *where* k *is fixed and equal to* $n - 1$ *or* n *or* $n + 1$. This is obvious if $x^2 = e$ for all x in G. Suppose that x in G has order $\bar{x} > 2$, and let $y \neq e$ be any other element in G with order $\bar{y}$. Case I. $\bar{y} \neq 2$. Then $\alpha = \alpha(x\,y)\,(x^{-1}y^{-1}) \in [X^{-1}] \cap [X^0] \subseteq [X^n]$. If $\bar{x} \mid n - 1$, then $y = x\,\alpha = x^n\,\alpha = (x\,\alpha)^n = y^n$, hence $\bar{y} \mid n - 1$. If $\bar{x} \mid n$, then $e = e\,\alpha = x^n\,\alpha = (x\,\alpha)^n = y^n$, hence $\bar{y} \mid n$. If $\bar{x} \mid n + 1$, then $y^{-1} = x^{-1}\,\alpha = x^n\,\alpha = (x\,\alpha)^n = y^n$, hence $\bar{y} \mid n + 1$. Case II. $\bar{y} = 2$. $\alpha = \alpha(e\,y) \in [X^{-1}] = [X^n]$. n is odd. Otherwise $y = e\,\alpha = y^n\,\alpha = (y\,\alpha)^n = e^n = e$. $\bar{x} \nmid n$. Otherwise $y = e\,\alpha = x^n\,\alpha = (x\,\alpha)^n = x^n = e$. Thus $\bar{x}$ divides $n - 1$ or $n + 1$ both of which are even, and hence is divisible by $\bar{y}$. *If* $x^{n-1} = e$ *for all* x *in* G, *then* $x^{n+1} = e$ *for all* x *in* G. For by Lemma 3.1, $[X] = [X^n] = [X^{-1}]$. By Theorem 3.1, $x^2 = e$ for all x in G. Thus $2 \mid n - 1$. Hence $x^{n+1} = e$ for all x in G. If $x^n = e$ for all x in G, then $|G| = 1$ or $|G| = 3$ and $3 \nmid n - 1$. For by Lemma 3.1, $[X^0] = [X^n] = [X^{-1}]$. By Theorem 3.3, $x^{-1} = e$ for all x in G or $|G| = 3$ and $3 \nmid n - 1$. This completes the proof of the theorem, and from the argument in the proof of Case II we have

Corollary I. *If* $[X^n] = [X^{-1}]$ *and* G *contains an element of order* 2, *then* n *is odd.*

Combining Theorems 3.1, 3.3 and 3.5 we have

Corollary II. *If* $|G| \neq 3$, *then the following are equivalent.* (a) $x^{n+1} = e$ *for all* x *in* G. (b) $[X^n] = [X^{-1}]$. (c) $[X^{n+1}] = [X^0]$. (d) $[X^{n+2}] = [X]$.

The proofs of Theorems 3.1, 3.3 and 3.5 are progressively harder because more care must be used in selecting elements from $[X^{-1}]$ than from $[X^0]$, and from $[X^0]$ than from $[X]$. In order to obtain more information of this type it is apparent that the structures of the groups $[X^n]$ must be investigated. This we do in the next section.

4. Structure of $[X^n]$. In order to determine the structure of the group $[X^n]$, where n is a fixed but arbitrary integer, the concept of an n-set is useful. Let S and T be subsets of G and let $g \in G$. S is an *n-set* (of G) if for every a in G, $a \in S$ if and only if $a^n \in S$. An n-set T is *maximal* if T is contained in no n-sets except T and G. T is *minimal* if T contains no n-sets except T and $\square$. Let G^g be the intersection of all n-sets that contain g, and let G_g be the join of all n-sets that do not contain g. The proofs of the following eight propositions are easy and most of them will be omitted or just outlined.

(1) *The set of all n-sets of G is closed with respect to intersection, joins and relative complements.*

(2) *S is a minimal n-set if and only if $G - S$ is a maximal n-set.*

For if S is a minimal n-set and $G - S \subseteq T$ an n-set, then $S \supseteq T \cap S$ an n-set. Thus $T \cap S = S$ or $T \cap S = \square$. If $T \cap S = S$, then $T \supseteq S \cup (G - S)$, hence $T = G$. If $T \cap S = \square$, then $T = G - S$. Therefore $G - S$ is a maximal n-set.

(3) *G^g is a minimal n-set, and $G - G^g = G_g$.* For if $G^g \supset T$ an n-set, then $G^g - T$ is an n-set that contains g. Thus $G^g - T \supseteq G^g$. Hence $T = \square$ and G^g is minimal. $G - G^g$ is a maximal n-set that does not contain g. Thus $G \supset G_g \supseteq G - G^g$, hence $G_g = G - G^g$.

(4) *If S is a minimal n-set, then $S = G^s$ for every s in S. Thus G is covered by the set of pairwise disjoint minimal n-sets G^g.*

(5) *For each a in G, $G^a = \{b \in G : b^{n^r} = a^{n^s}$ for some non-negative integers $r, s\}$. In particular, $G^e = \{b \in G : b^{n^r} = e$ for some $r \geqslant 0\}$. Therefore G contains no proper n-sets if and only if for each g in G there exists an $m \geqslant 0$ such that $g^{n^m} = e$.* For let $A = \{b \in G: b^{n^r} = a^{n^s}$ for some non-negative integers $r, s\}$. By simple induction arguments $G^a \supseteq A$. Also $a \in A$. Thus it sufficies to show that A is an n-set. If $b \in A$, $b^{n^r} = a^{n^s}$ for $r \geqslant 0 \leqslant s$. Hence $(b^n)^{n^r} = b^{n^{r+1}} = a^{n^{s+1}}$, and $b^n \in A$. Conversely if $b^n \in A$, then $b^{n^{r+1}} = (b^n)^{n^r} = a^{n^s}$ for some $r \geqslant 0 \leqslant s$, hence $b \in A$.

(6) *If S is an n-set and $\alpha \in [X^n]_S$ (that is, α is a permutation of S such that $s^n \alpha = (s\alpha)^n$ for all s in S), then $\alpha^* \in [X^n]$, where $g\alpha^* = g\alpha$ if $g \in S$ and $g\alpha^* = g$ if $g \in G \smallsetminus S$.* Note that Lemma 3.2 is a special case of this proposition.

(7) *If $\alpha \in [X^n]$, then S is an n-set (minimal or maximal n-set) if and only if $S\alpha$ is an n-set (minimal or maximal n-set). $(G^g)\alpha = G^{g\alpha}$ and $(G_g)\alpha = G_{g\alpha}$ for every g in G.* For suppose that S is an n-set. If $x \in S\alpha$, then $x = y\alpha$ where $y \in S$. Thus $y^n \in S$, and $x^n = (y\alpha)^n = y^n\alpha \in S\alpha$. If $x^n \in S\alpha$, then $(x\alpha^{-1})^n = x^n\alpha^{-1} \in S$. Thus $x\alpha^{-1} \in S$, and $x = (x\alpha^{-1})\alpha \in S\alpha$. Therefore S is an n-set. If S is a minimal n-set and $S\alpha \supset T$ an n-set, then $S = (S\alpha)\alpha^{-1} \supset T\alpha^{-1}$ and $T\alpha^{-1}$ is an n-set. Thus since S is minimal, $T\alpha^{-1} = \square = T$, and $S\alpha$ is minimal. If S is a maximal n-set, then $G \smallsetminus S$ is a minimal n-set. Hence $(G \smallsetminus S)\alpha = G \smallsetminus S\alpha$ is a minimal n-set. Thus $S\alpha$ is maximal.

An immediate consequence of (7) is that each α in $[X^n]$ induces a permutation on the set H *of all the minimal n-sets G^g of G*. Thus H is split into transitivity sets H_γ by $[X^n]$. That is, $H = \bigcup_{\gamma\in\Gamma} H_\gamma$, where $H_\alpha \cap H_\beta = \square$ unless $\alpha = \beta$, and $G^a, G^b \in H_\gamma$ if and only if there exists a σ in $[X^n]$ such that $(G^a)\sigma = G^b$. Two n-sets are *isomorphic* if there exists a 1-1 mapping τ of S onto T such that $s^n\tau = (s\tau)^n$ for all s in S.

(8) *If* $G^g, G^h \in H_\gamma$, *then* G^g *and* G^h *are isomorphic. If* S *and* T *are isomorphic n-sets of* G, *then the corresponding groups* $[X^n]_S$ *and* $[X^n]_T$ *are isomorphic.* For let τ be an isomorphism of S onto T, and consider α in $[X^n]_S$. Then $\alpha\sigma = \tau^{-1}\alpha\tau \in [X^n]_T$ and it is easy to verify that σ is an isomorphism of $[X^n]_S$ onto $[X^n]_T$.

For each γ in Γ let $\mathscr{H}_\gamma$ be the join of all the G^g in H_γ, and let $[X^n]_\gamma$ be the group of all permutations α of $\mathscr{H}_\gamma$ such that $x^n\alpha = (x\alpha)^n$ for all x in $\mathscr{H}_\gamma$.

THEOREM 4.1. $[X^n]$ *is isomorphic to the unrestricted direct product* $\mathbf{P}[X^n]_\gamma$ *of all the* $[X^n]_\gamma$.

PROOF. Consider γ in Γ and α in $[X^n]$. α induces a permutation α_γ of $\mathscr{H}_\gamma$. It follows from (6) that the mapping $\alpha \to \alpha_\gamma$ is a homomorphism of $[X^n]$ onto $[X^n]_\gamma$. Define $\alpha\tau = \{\alpha_\gamma : \gamma \in \Gamma\}$. Then it follows easily that τ is the desired isomorphism.

We next investigate the structures of the groups $[X^n]_\gamma$. By (8) all the G^g in H_γ are isomorphic, and the corresponding groups $[X^n]_{G^g}$ are isomorphic. Denote the elements in H_γ by U, V and W. Let U be a fixed n-set in H_γ and $\mathscr{U} = [X^n]_U$. For each V in H_γ let τ_V be an isomorphism of U into V. For each α in $[X^n]$ and each V in H_γ, let $\tilde{\alpha}_V$ be the isomorphism of V onto $V\alpha$ that is induced by α. Let $\alpha_V = \tau_V\,\tilde{\alpha}_V\,(\tau_{V\alpha})^{-1}$. Then α_V is an automorphism of U. That is, $\alpha_V \in \mathscr{U}$. For α in $[X^n]_\gamma$ we define

$$\alpha\sigma = (\{\alpha_V : V \in H_\gamma\}, \bar{\alpha}) = (\{\alpha_V\}, \bar{\alpha}),$$

where $\bar{\alpha}$ is the permutation of H_γ induced by α. Let $\bar{H}_\gamma$ be the group of all permutations of H_γ and let $\mathbf{P}\,\mathscr{U}_V$ be the large direct product of the groups $\mathscr{U}_V$, where $\mathscr{U}_V = \mathscr{U}$ for all V in H_γ. Let $B_\gamma = (\mathbf{P}\mathscr{U}_V) \times \bar{H}_\gamma$. For $(\{\alpha_V\}\bar{\alpha})$, and $(\{\beta_V\}, \bar{\beta})$ in B_γ we define $(\{\alpha_V\}, \bar{\alpha})(\{\beta_V\}, \bar{\beta}) = (\{(\alpha_V\,\beta_{V\bar{\alpha}})_V\}, \bar{\alpha}\bar{\beta})$.

THEOREM 4.2. B_γ *is a group and* σ *is an isomorphism of* $[X^n]_\gamma$ *onto* B_γ.

PROOF. Consider α and β in $[X^n]_\gamma$. $(\alpha\beta)\sigma = (\{(\alpha\beta)_V\}, \overline{\alpha\beta})$ and $\alpha\sigma\beta\sigma = (\{\alpha_V\}, \bar{\alpha})(\{\beta_V\}, \bar{\beta}) = (\{\alpha_V\,\beta_{V\bar{\alpha}})_V\}, \bar{\alpha}\bar{\beta})$. Clearly $\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$.

Pick x in U. $x(\alpha\,\beta)_V = x\,\tau_V\,\widetilde{\alpha\beta}_V\,(\tau_{V\alpha\beta})^{-1}$. $x\,\alpha_V\,\beta_{V\bar{\alpha}} = (x\,\alpha_V)\,\beta_{V\bar{\alpha}} = [x\,\tau_V\,\tilde{\alpha}_V\,(\tau_{V\alpha})^{-1}]\,\beta_{V\alpha} = x\,\tau_V\,\tilde{\alpha}_V\,(\tau_{V\alpha})^{-1}\,\tau_{V\alpha}\,\beta_{V\alpha}\,(\tau_{V\alpha\beta})^{-1} = x\,\tau_V\,\tilde{\alpha}_V\,\tilde{\beta}_{V\alpha}\,(\tau_{V\alpha\beta})^{-1}$. But $\widetilde{\alpha\beta}_V = \tilde{\alpha}_V\,\tilde{\beta}_{V\alpha}$. Thus σ is a homomorphism. Consider $(\{\alpha_V\}, \bar{\alpha})$ in B_γ. For each α_V define $\tilde{\alpha}_V = \tau_V^{-1}\,\alpha_V\,\tau_{V\bar{\alpha}}$. Then $\tilde{\alpha}_V$ is an isomorphism of V onto V_α. Each g in $\mathscr{H}_\gamma$ belongs to a unique $V \in H_\gamma$. Define $g\,\tilde{\alpha} = g\,\tilde{\alpha}_V$. Then $\alpha\,\sigma = (\{\alpha_V\}, \bar{\alpha})$. Thus σ is a homomorphism of $[X^n]_\gamma$ onto B_γ, hence B_γ is a group. If $\alpha\sigma = \theta$, then $\bar{\alpha} = \theta$ and $\tilde{\alpha}_V = \theta$ for all V in H_γ. Thus $\alpha = \theta$, and σ is an isomorphism.

Corollary I. *$[X^n]_\gamma$ is determined to within an isomorphism by $|H_\gamma|$ and the group $\mathscr{U}$, where $\mathscr{U} \simeq [X^n]_{G^g}$ for each G^g in H.*

Thus we have determined the structure of $[X^n]$ modulo the structures of the $[X_n]_{G^g}$. $A_\gamma = (\mathbf{P}\,\mathscr{U}_V) \times \theta$ is a normal subgroup of B_γ. Let $\mathfrak{A}_\gamma = A_\gamma\,\sigma^{-1} = \{\alpha \in [X^n]_\gamma : G^g\,\alpha = G^g \text{ for all } G^g \text{ in } H_\gamma\}$. $C_\gamma = \theta \times \bar{H}_\gamma$ is a subgroup of G and the mapping of $(\{\alpha_\gamma\}, \bar{\alpha})$ in B_γ upon $\bar{\alpha}$ in $\bar{H}_\gamma$ is a homomorphism of B_γ onto $\bar{H}_\gamma$ with kernel A_γ.

Corollary II. *$[X^n]_\gamma$ is a normal splitting extension of $\mathfrak{A}_\gamma$ by $\bar{H}_\gamma$.*

$[X_n] = \mathbf{P}\,[X^n]_\gamma \simeq \mathbf{P}\,B_\gamma = \mathbf{P}\,A_\gamma\,C_\gamma \simeq (\mathbf{P}\,A_\gamma)\,(\mathbf{P}\,C_\gamma)$. $\mathbf{P}\,C_\gamma \simeq \mathbf{P}\,\bar{H}_\gamma$ and $\mathbf{P}\,A_\gamma \simeq \mathbf{P}\,[X^n]_{G^g} = \{\alpha \in [X^n] : G^g\,\alpha = G^g \text{ for all } g \text{ in } G\} = \mathfrak{A}$.

Corollary III. *$[X^n]$ is a normal splitting extension of $\mathfrak{A}$ by $\mathbf{P}\,\bar{H}_\gamma$.*

Remarks. The structure of $[X^n]$ can be determined in a much more general setting. Let $\mathfrak{Y}$ be a set and let $\mathfrak{n}$ be a mapping of $\mathfrak{Y}$ into $\mathfrak{Y}$. Let $[\mathfrak{X}^n]$ be the group of all permutations α of $\mathfrak{Y}$ such that $\mathfrak{n}\,\alpha = \alpha\mathfrak{n}$. A subset $\mathfrak{S}$ of $\mathfrak{Y}$ is an $\mathfrak{n}$-set if for every a in $\mathfrak{Y}$, $a \in \mathfrak{S}$ if and only if $a^{\mathfrak{n}} \in \mathfrak{S}$. Then all of the preceding theory goes through (except for the remarks about G^e). Theorem 4.2 is the special case where $\mathfrak{Y}$ is a group and $a^{\mathfrak{n}} = a^n$ for all $a \in \mathfrak{Y}$. If n is a positive integer, then we have a theorem about semigroups. Similarly, most of the following theory can be proven in this generality. For example, the generalization of condition $[n]$ is the requirement that $\mathfrak{n}$ is 1-1. The generalization of $[X^{-1}]$ is the requirement that $\mathfrak{n}^2$ is the identity permutation of $\mathfrak{Y}$.

For g, h in G define $g \leqslant h$ if $g^{n^r} = h$ for some integer $r \geqslant 0$. Then $\leqslant$ is a transitive relation, but it is not a partial order. By (5) G_a consists of all elements of G that are less than or equal to one of $a, a^n, a^{n^2}, \ldots$.

LEMMA 4.1. *If G^g contains a least element, and $x^n = y^n$ implies $x = y$ for all x and y in b^g, then $[X^n]_{G^g}$ is a finite cyclic group.*

PROOF. Let a be a least element of G^g, then $G^g = a, a^n, a^{n^2}, \ldots$. Let $\alpha \in [X^n]_{G^g}$. Then clearly α is determined by $a\alpha$. In particular, $\alpha = \theta$ if and only if $a\alpha = a$. Suppose that $\alpha \neq \theta$. $a\alpha = a^{n^p} \neq a$, and $a = a^{n^q}\alpha = (a\alpha)^{n^q} = a^{n^{p+q}}$. Thus the a^{n^i} are not distinct. Let a^{n^r} be the first one that is equal to one of the preceding ones, say a^{n^s}. Let m be the order of a. Then $m \mid n^{p+q} - 1$ and $m \mid n^s(n^{r-s} - 1)$. Since $(n^{p+q} - 1, n^s) = 1$, $m \mid n^{r-s} - 1$. Therefore $a^{n^{r-s}} = a$. By the minimality of r, $s = 0$. It is now easy to verify that $[X^n]_{G^g}$ is a cyclic group of order r that is generated by $\alpha : a\alpha = a^n$.

COROLLARY. *Suppose that G^g satisfies the hypothesis of the lemma. If $[X^n]_{G^g} \neq \theta$, then G^g is finite and $[X^n]_{G^g}$ is a finite cyclic group of order $|G^g|$.*

We next determine the structure of $\mathfrak{A} = \mathbf{P}\,[X^n]_{G^g}$ for a restricted class of groups. *Suppose that G satisfies :*

$$\textit{If } a^n = b^n, \textit{ then } a = b \textit{ for all } a, b \textit{ in } G. \qquad [n]$$

In particular, no element of G has order that divides a positive power of n (except e). Thus by (5) $G^e = G$. Once again consider G^g. First suppose that G^g contains a least element a. $G^g = a, a^n, a^{n^2}, \ldots$. If G^g is infinite, then all of the a^{n^i} are distinct and by the corollary to Lemma 4.1, $[X^n]_{G^g} = \theta$. Let $G^h = b, b^n, b^{n^2}, \ldots$ be another minimal n-set of this type. The mapping that interchanges a^{n^r} with b^{n^r} for all $r \geqslant 0$ and leaves the other elements of G fixed belongs to $[X^n]$. Thus $H_0 = \{G^g : G^g$ is infinite and contains a least element$\}$ is a transitivity set of H with respect to $[X^n]$.

Next suppose that G^g is finite and contains a least element a. Let a^{n^r} be the first a^{n^i} that is equal to one of the preceding ones. It follows at once from $[n]$ that $a^{n^r} = a$. Thus by the proof of

Lemma 1.4, $[X^n]_{G^g}$ is a finite cyclic group of order r. For $i = 1, 2, \ldots$ let $H_i = \{G^g : G^g$ has a least element and $|G^g| = i\}$. Then as above the H_i are transitivity sets with respect to $[X^n]$.

Finally suppose that G^g contains no least element. For each h in G^g, $x^n = h$ has a solution in G^g. For if $x^n = h$ has no solution, then it follows from $[n]$ that $G^g = h, h^n, h^{n^2}, \ldots$. Thus h is a least element of G^g, a contradiction. Denote the *unique* solution of $x^{n^i} = h$ by h^{1/n^i}. Then $G^g = \ldots, g^{1/n^2}, g^{1/n}, g, g^n, g^{n^2}, \ldots$, where all of the elements are distinct. It now follows that $[X^n]_{G^g}$ is an infinite cyclic group that is generated by $\alpha : y\alpha = y^n$ for all y in G^g. Let $H_\infty = \{G^g : G^g$ contains no least element$\}$. Then by a straightforward argument H_∞ is a transitivity set with respect to $[X^n]$. $H = H_0 \cup H_1 \cup H_2 \cup \ldots \cup H_\infty$.

THEOREM 4.3. *If G satisfies $[n]$, then $\mathfrak{A}$ is isomorphic to the unrestricted direct product of $|H_i|$ finite cyclic groups of order i for $i = 2, 3, \ldots$ and $|H_\infty|$ infinite cyclic groups.*

COROLLARY I. *Suppose that G satisfies $[n]$. Then $G^g \in H_0$ if and only if every element of G^g has finite height with respect to n.*

PROOF. If $G^g \in H_0$, then $G^g = a, a^n, a^{n^2}, \ldots$, and a^{n^r} has height r with respect to n. Conversely suppose that every element in G^g has finite height with respect to n. Then there exists an element a in G^g such that $x^n = a$ has no solution in G^g. Thus $G^g = a, a^n, a^{n^2}, \ldots$. If G^g is finite, then $a = a^{n^r}$ for some $r > 0$, hence $x^n = a$ has a solution. Thus G^g is infinite and belongs to H_0.

COROLLARY II. *If G satisfies $[n]$ and every $g \neq e$ in G has finite height with respect to n, then $[X^n]$ is isomorphic to the group of all permutations of $L_n = \{g \in G : x^n = g$ has no solution in $G\}$.*

For in this case $[X^n] = [X^n]_0 \simeq \bar{H}_0 \simeq L_n$. For example, if G is the group of integers with respect to addition, then L_n is the set of all integers not divisible by n. Thus $[nX] \simeq [X]$ for all $n \neq -1$.

COROLLARY III. *Suppose that G satisfies $[n]$ and that for every g in G, $g^{n^m} = g$ if and only if $m = 0$ or $g = e$. Then $\mathfrak{A}$ is isomorphic*

to the unrestricted direct product of $|H_\infty|$ *infinite cyclic groups and* $[X^n]/\mathfrak{A}$ *is isomorphic to* $\bar{H}_0 \otimes \bar{H}_\infty$.

For in this case each G^g is infinite unless $g = e$. Thus $H = H_0 \cup H_\infty \cup \{G^e\}$.

LEMMA 4.2. *If* $S \neq \theta$ *is a normal subgroup of* $[X]$, *then* S *contains the alternating subgroup* A *of* $\{\alpha \in [X] : g\alpha = g$ *for all but a finite number of* g *in* $G\}$ *provided that* $|G| \neq 4$. *If* $|G| = 4$, *then* S *contains the normal four group in* A.

The proof for finite G is well known, and a slight modification of this proof takes care of the infinite case. In the following we shall refer to A as the alternating subgroup of $[X]$.

COROLLARY. *The center of* $[X]$ *is the identity provided that* $|G| \neq 2$. $[X]$ *is directly indecomposable.*

If G is the four group or a group of order 1, 2 or 3, then $[X^m] = [X^0]$ or $[X^m] = [X]$ for every integer m. Thus $[X^m]$ is a symmetric group. If G is a cyclic group of order 4, then $[X^m]$ is symmetric unless $m - 1$ is even and not divisible by 4. In the latter case $[X^m] = [X^{-1}] =$ the four group. These results follow from the theorems in the last section and some easy computations.

THEOREM 4.4. *If* $|G| \neq 3$ *and* G *satisfies* $[n]$, *then* $[X^n]$ *is a symmetric group if and only if* G *satisfies one of the following.* (a) $g^n = g$ *for all* g *in* G. (b) *Every* g *in* G *has finite height with respect to* n.

PROOF. If G satisfies (a), then $G^g = g$ for all g in G. Thus $H = H_1$ and $[X^n] \simeq \bar{H}_1 \simeq [X]$. In fact, $[X^n] = [X]$ by Theorem 3.1. If G satisfies (b), then by Corollary I of Theorem 4.3, $H = H_0 \cup \{G^e\}$. Thus $[X^n] \simeq \bar{H}_0$. Conversely suppose that $[X^n]$ is a symmetric group. Assume by way of contradiction, that $\mathfrak{A} \neq \theta$. Then since $\mathfrak{A}$ is a normal abelian subgroup of $[X^n]$, $[X^n] = S_2$, S_3 or S_4 (Lemma 4.2). If $[X^n] = S_2$, then $\mathfrak{A} = [X^n]$, and it follows by an easy argument that $H = \{G^e\} \cup H_2$ and $|H_2| = 1$. Thus $|G| = 3$, a contradiction. If $[X^n] = S_3$, then $\mathfrak{A} \simeq A_3$ and $\mathbf{P}\,\bar{H}_\gamma \simeq [X^n]/\mathfrak{A} \simeq S_2$. Thus $|H_3| = 1$, $|H_\alpha| = 0$ for all $\alpha \neq 0$, 1, 3, and there exists a β such that $|H_\beta| = 2$ and $|H_\alpha| \leqslant 1$ for all

$\alpha \neq \beta$. Thus $[X^n] = [X^n]_3 \otimes [X^n]_\beta$, a contradiction. If $[X^n] = S_4$, then $\mathfrak{A}$ is the four group and $S_3 = [X^n]/\mathfrak{A} \simeq \mathbf{P}\,\overline{H}_\gamma$. Thus there exists a γ such that $|H_\gamma| = 3$ and $|H_\alpha| \leqslant 1$ for all $\alpha \neq \gamma$. But since $\mathfrak{A}$ is the four group, $|H_2| = 2$, a contradiction. Therefore $\mathfrak{A} = \theta$.

It follows that $H = H_0 \cup H_1$ and $G^e \in H_1$. If $H = H_1$, then G satisfies (a). If $H_0 \neq \square$, then there exists an element g in G such that $G^g \in H_0$. Also $G^{g^{-1}} \in H_0$, hence $|H_0| \geqslant 2$. But $[X^n] \simeq \overline{H}_0 \otimes \overline{H}_1$ and $[X^n]$ is directly indecomposable. Thus $\overline{H}_1 = e$, and $H = H_0 \cup \{e\}$. Therefore by Corollary I of Theorem 4.3, G satisfies (b).

Note that if $|G| = 3$ and $n = -1$, then $[X^{-1}] = [X^0]$, but that G does not satisfy (a) or (b). Thus the hypothesis that $|G| \neq 3$ in Theorem 4.4 can not be removed.

For the remainder of this section we investigate the structure of $[X^{-1}]$. Clearly G satisfies $[-1]$, and every element in G has infinite height with respect to -1. Thus it follows from Theorem 4.4 that if $|G| \neq 3$, then $[X^{-1}]$ is symmetric if and only if $g^2 = e$ for all g in G. For each g in G, $G^g = \{g, g^{-1}\}$. Therefore $H = H_1 \cup H_2$, and from the preceding theorem we have

THEOREM 4.5. $[X^{-1}] \simeq [X^{-1}]_1 \otimes [X^{-1}]_2$. $[X^{-1}]_1 \simeq H_1$. $[X^{-1}]_2 \simeq (\mathbf{P}\,\mathfrak{U}_V) \times \overline{H}_2$, *where* $\mathfrak{U}_V = \mathfrak{U}$ *for all* V in H_2, $\mathfrak{U}$ *is a cyclic group of order* 2, *and* $(\{\alpha_V\}, \alpha)\ (\{\beta_V\}, \beta) = (\{(\alpha_V\, \beta_{V\alpha})_V\}, \alpha\beta)$.

Let $T = \{g \in G: g^2 = e\}$, $P = \{\alpha \in [X^{-1}]: \alpha = \theta$ on $T\}$ and $Q = \{\alpha \in [X^{-1}]: \alpha = \theta$ on $G \setminus T\}$.

COROLLARY. $[X^{-1}] = P \otimes Q$, $Q \simeq [X^{-1}]_1$, $P \simeq [X^{-1}]_2$, *and* $[X^{-1}]$ *is determined to within an isomorphism by* $|T|$ *and* $|G \setminus T|$.

In particular, Q is directly indecomposable. In order to determine whether or not P is directly indecomposable we need the following lemma. If w is an element of a group W, then $\langle w \rangle$ will denote the cyclic subgroup generated by w. In particular $\langle \pi \rangle$ is a subgroup of order 2 of $[X^{-1}]$. Denote the center of W by $Z(W)$.

LEMMA 4.3. $Z(P) = \langle \pi \rangle$. $Z([X^{-1}]) = \langle \pi \rangle$ *if and only if* $|T| \neq 2$, *If* $|T| = 2$ *then* $Z([X^{-1}]) = \langle \pi \rangle \otimes Q$.

PROOF. Clearly $Z(P) \supseteq \langle\pi\rangle$. Suppose that $\alpha \in Z(P) \smallsetminus \langle\pi\rangle$. Then there exists g in $G \smallsetminus T$ such that $g\alpha \neq g$ or g^{-1}. Thus $g\alpha \neq g\alpha^2$ or $(g\alpha^2)^{-1}$. It is easy to construct a β in P such that $g\beta = g\alpha$ and $g\alpha\beta = (g\alpha^2)^{-1}$. But then $g\alpha^2 = g\beta\alpha = g\alpha\beta = (g\alpha^2)^{-1}$. Thus $g\alpha^2 \in T$, hence $g \in T$, a contradiction. $Z([X^{-1}]) = (Z([X^{-1}] \cap P) \otimes (Z([X^{-1}]) \cap Q) = Z(P) \otimes Z(Q) = \langle\pi\rangle \otimes Z(Q)$. By the corollary to Lemma 4.2, $Z(Q) = \theta$ unless $|T| = 2$.

THEOREM 4.6. *Either P is directly indecomposable or $P = \langle\pi\rangle \otimes N$, where N is directly indecomopsable. P is directly decomposable if and only if $|G \smallsetminus T|$ is finite, greater than 2 and not divisible by 4.*

PROOF. Let $\mathfrak{A} = \{\alpha \in P : G^g\alpha = G^g$ for all G^g in $H_2\}$. Pick a subset K_2 of G such that K_2 contains one and only one element from each G^g in H_2. Let $\mathfrak{C} = \{\alpha \in P : K_2\alpha = K_2\}$. Then $P = \mathfrak{A}\mathfrak{C}$, $\mathfrak{A} \cap \mathfrak{C} = \theta$, $\mathfrak{C}$ is a symmetric group, and $\mathfrak{A}$ is an abelian normal subgroup of P all of whose elements have order 2. Suppose that $|G \smallsetminus T|$ is finite, greater than 2 and not divisable by 4. Then $|H_2| = (1/2)|G \smallsetminus T|$ is odd. Let $\mathfrak{A}' = \{\alpha \in \mathfrak{A} : \alpha$ interchanges an even number of pairs x, x^{-1} in $G \smallsetminus T\}$. Then $\mathfrak{A}'$ is a group and $\mathfrak{A} = \langle\pi\rangle \otimes \mathfrak{A}'$. Moreover, if $\gamma \in \mathfrak{C}$ and $\alpha \in \mathfrak{A}'$, then $\gamma\alpha\gamma^{-1} \in \mathfrak{A}'$, since $\gamma\alpha\gamma^{-1}$ interchanges x, x^{-1} if and only if α interchanges $x\gamma$, $(x\gamma)^{-1}$. Therefore $\mathfrak{A}'$ is normal in P and $P = \langle\pi\rangle \otimes \mathfrak{A}'\mathfrak{C}$. Since $|G \smallsetminus T| > 2$, $\langle\pi\rangle \neq \theta \neq \mathfrak{A}'\mathfrak{C}$.

If $P = M \otimes N$, where M and N are non-trivial subgroups of P, then $M = \langle\pi\rangle$ or $N = \langle\pi\rangle$. First assume that $|G \smallsetminus T| > 8$. Then the degree of $\mathfrak{C} = |H_2| > 4$. Let μ be the projection of P onto $\mathfrak{C}$ with kernel $\mathfrak{A}$. If $x, y \in M_\mu \cap N_\mu$, then $x = a\mu$ and $y = b\mu$ for some a in M and b in N. $xy = a\mu b\mu = ab\mu = ba\mu = b\mu a\mu = yx$. Therefore $M\mu \cap N\mu$ is an abelian subgroup of $\mathfrak{C}$. If $M\mu \neq \theta \neq N\mu$, then by Lemma 4.2, $M\mu \cap N\mu$ contains the non-abelian alternating subgroup of $\mathfrak{C}$, a contradiction. Thus $M\mu = \theta$ or $N\mu = \theta$. If $M\mu = \theta$, then $M \subseteq \mathfrak{A}$. Thus M is abelian and $\theta \subset M \subseteq Z(P) = \langle\pi\rangle$.

Therefore $M = \langle\pi\rangle$. Similarly if $N\mu = \theta$, then $N = \langle\pi\rangle$. Next assume that $|G \smallsetminus T| \leqslant 8$. It suffices to show that $|M| \leqslant 5$ or $|N| \leqslant 5$.

For if $|M| \leqslant 5$ then M is abelian, hence $\theta \subset M \subseteq Z(P) = \langle \pi \rangle$. $|P| = |\mathfrak{A}|\,|\mathfrak{C}| = (2^{(|G \sim T|)/2})\,([(G \sim T)/2]!)$. If $|G \sim T| \leqslant 4$, then $|P| \leqslant 8$, hence $|M| \leqslant 4$ or $|N| \leqslant 4$. If $|G \sim T| = 6$, then $P = \langle \pi \rangle \otimes \mathfrak{A}'\mathfrak{C}$ and $|\mathfrak{A}'\mathfrak{C}| = 24$. Hence if $\mathfrak{A}'\mathfrak{C}$ is directly decomposable, then one of the factors is abelian. But this means that $Z(P) \neq \langle \pi \rangle$. Therefore $\mathfrak{A}'\mathfrak{C}$ is indecomposable and by the Krull Schmidt theorem $|M| = 2$ or $|N| = 2$. Now suppose that $|G \sim T| = 8$. Then one of the factors, say N, contains all of the elements in P of order 3 (since $|P|$ is divisible by 3 but not by 9). Thus N contains the alternating subgroup of C for all choices of K_2. It follows that $|N| > 48$. Hence $|M| \leqslant 4$.

If $P = \langle \pi \rangle \otimes N$ and $N = N_1 \otimes N_2$, where N_1 and N_2 are non-trivial subgroups of P, then $\langle \pi \rangle = \langle \pi \rangle \otimes N_1$ or $\langle \pi \rangle = N_2$, a contradiction in both cases. Therefore if $P = \langle \pi \rangle \otimes N$, then N is directly indecomposable.

Finally suppose that P is directly decomposable. Then $P = \langle \pi \rangle \otimes N$ and $|G \sim T| > 2$. $\mathfrak{A} = \langle \pi \rangle \otimes (\mathfrak{A} \cap N)$, and $\mathfrak{A} \cap N$ is a normal subgroup of P. If $\alpha, \beta \in \mathfrak{A}$ and $\alpha\beta = \pi$, then one but not both of α and β belong to $\mathfrak{A} \cap N$. For if $\alpha, \beta \in \mathfrak{A} \cap N$, then $\alpha = \pi\alpha'$ and $\beta = \pi\beta'$, where $\alpha', \beta' \in \mathfrak{A} \cap N$, hence $\pi = \alpha\beta = \alpha'\beta' \in \mathfrak{A} \cap N$. Suppose that $|G \sim T|$ is infinite or finite and divisible by 4. Then we can pick a pair α, β in $\mathfrak{A}$ such that $\alpha\beta = \pi$, α and β both interchange the same number of pairs x, x^{-1} in $G \sim T$, and $x\alpha = x^{-1}$ if and only if $x\beta = x$ for all x in $G \sim T$. But then there exists a $\gamma \in \mathfrak{C}$ such that $\gamma\alpha\gamma^{-1} = \beta$. Thus both α and β belong to $\mathfrak{A} \cap N$ or neither belong to $\mathfrak{A} \cap N$, a contradiction. This completes the proof of Theorem 4.6.

Thus $[X^{-1}]$ is the direct product of 1, 2 or 3 non-trivial directly indecomposable groups. Suppose that G contains no elements of order 2. Then $[X^{-1}] = P$ is directly indecomposable if and only if $|G|$ is infinite or $|G| - 1$ is finite and equal to 2 or divisible by 4. *For the remainder of this section assume that* $|T| > 4$.

COROLLARY I. *Suppose that* $[X^{-1}] = A \otimes B \otimes C$, *where* A, B *and* C *are non-trivial subgroups of* $[X^{-1}]$. *Then for a suitable ordering of*

the factors $A = \langle \pi \rangle$, $\langle \pi \rangle \otimes B = P$ *and* $C \simeq Q$. *In particular* A, B *and* C *are directly indecomposable. Thus* $[X^{-1}]$ *is the direct product of at most* 3 *non-trivial subgroups.*

PROOF. Let ϵ_1 and ϵ_2 be the projections of $[X^{-1}]$ onto Q and P respectively. Two of the triple $A\epsilon_1$, $B\epsilon_1$, $C\epsilon_1$ must equal θ. For if $A\epsilon_1 \neq \theta \neq B\epsilon_1$ then $A\epsilon_1 \cap B\epsilon_1$, is an abelian subgroup of Q that contains the alternating subgroup of Q. Thus, without loss of generality, $A \subseteq P$ and $B \subset P$. Each p in P has a unique representtion $p = abc$, where $a \in A$, $b \in B$ and $c \in C$. $abc = p = p\epsilon_2 = ab(c\epsilon_2)$, hence $c = c\epsilon_2$. Thus $c \in C \cap P$ and $P = A \otimes B \otimes (P \cap C)$. By Theorem 4.6, $P \cap C = \theta$ and either $A = \langle \pi \rangle$ or $B = \langle \pi \rangle$. Therefore $P = A \otimes B$ and $Q \simeq [X^{-1}]/P = [X^{-1}]/(A \otimes B) \simeq C$.

COROLLARY II. $[X^{-1}] = A \otimes B \otimes C$, *where* A, B, C *are non-trivial groups if and only if the number of elements in* G *with order greater than* 2 *is finite, greater than* 2 *and not divisible by* 4.

COROLLARY III. *Suppose that* $[X^{-1}] = A \otimes B$, *where* A *and* B *are non-trivial directly indecomposable groups. Then for a suitable ordering of the factors* $A = P$ *and* $B \simeq Q$.

PROOF. Let ϵ_1 and ϵ_2 be as above. $A\epsilon_1 = \theta$ or $B\epsilon_1 = \theta$. Thus without loss of generality, $A \subseteq P$. Each p in P has a unique representation $p = ab$, where $a \in A$ and $b \in B$. $ab = p = p\epsilon_2 = a(b\epsilon_2)$. Thus $b \in P \cap B$ and $P = A \otimes (B \cap P)$. $A \otimes B = [X^{-1}] = P \otimes Q = A \otimes (B \cap P) \otimes Q$. $B \simeq [X^{-1}]/A = (B \cap P) \otimes Q$. Since B is directly indecomposable, $B \cap P = \theta$. Thus $P = A$ and $B \simeq Q$.

Thus $[X^{-1}]$ is the direct sum of at most 3 directly indecomposable factors, and any such factorization of $[X^{-1}]$ determines isomorphic images of P and Q. In fact, if P is directly indecomposable, then it is one of the factors, and if P is decomposable, then it is the sum of two of the factors.

5. The holomorph of G and related groups in $[X]$. For each g in G define $g^* : xg^* = xg$ for all x in G. Then $g^* \in [X]$ and the mapping ϕ of g in G onto g^* is an isomorphism of G into $[X]$.

Thus $S\phi$ is a subgroup of $[X]$ for each subgroup S of G. Denote $S\phi$ by S^*. $[XY^{-1}Z] = [XY]G^*$, hence $[XY^{-1}Z]$ is the holomorph of G. For $xg^*(yg^*)^{-1}\ zg^* = xgg^{-1}y^{-1}zg = xy^{-1}zg^*$ for all x, y, z, g in G. Thus $G^* \leqslant [XY^{-1}Z]$. Consider $\alpha \in [XY^{-1}Z]$, and let $\beta = \alpha((e\alpha)^{-1})^*$. Then $\beta \in [X^0] \cap [XY^{-1}Z]$ and for x, z in G, $(xz)\beta = (xe^{-1}z)\ \beta = x\beta(e\beta)^{-1}z\beta = x\beta z\beta$. Thus $\alpha = \beta(e\alpha)^* \in [XY]G^*$. Therefore $[XY^{-1}Z]$ is the normalizer of G^* in $[X]$, $[XY^{-1}Z]$ is a normal splitting extension of G^* by $[XY]$, every automorphism of G^* is induced by an inner automorphism of $[XY^{-1}Z]$ and $[XY^{-1}Z]$ contains all of the left multiplications by elements in G.

THEOREM 5.1. *For f in F the following are equivalent.*

(a) $[f] \supseteq [XY^{-1}Z]$ *for all* G.

(b) $[f] \supseteq [XY^{-1}Z]$ *when* $G = F$.

(c) $\prod_{i=1}^{2n+1} X_i^{(-1)^{i+1}} = f$.

PROOF. Clearly (a) implies (b), and it is easy to verify that $[XY^{-1}Z] = [XY]G^* \subseteq [\prod_{i=1}^{2n+1} X_i^{(-1)^{i+1}}]$. Hence (c) implies (a).

Finally suppose that (b) is true. Then clearly $f \neq X^0$. Thus f has an irreducible representation $f = \prod_{i=1}^{m} X_i^{e_i}$. Pick an X in $I \sim \{X_i\}_{i=1}^{m}$. $(\prod X_i^{e_i})X = (\prod X_i^{e_i})\ X^* = \prod(X_i X^*)^{e_i} = \prod(X_i X)^{e_i}$. The left hand side has length $m+1$ and is irreducible. Each simple reduction reduces the length of an element by 2. Thus when the right hand side is reduced, it will have an even length $m+1$. Therefore m is odd. If $e_1 = -1$, then $(X_1^{-1}\prod_{i=2}^{m} X_i^{e_i})X = X^{-1}X_1^{-1}\prod_{i=2}^{m}(X_i X)^{e_i}$ and no reduction will put X_1^{-1} on the left end of the right side. Thus $e_1 = 1$. If $e_2 = 1$, then $(X_2\prod_{i=3}^{m} X_i^{e_i})X = XX_2X\prod_{i=3}^{m}(X_iX)^{e_i}$, and no reduction will put X_2 on the left end of the right side. Thus $e_2 = -1$. Continuing in this way it follows that $e_i = (-1)^{i+1}$.

COROLLARY I. *If* $f = \prod_{i=1}^{2n+1} X_i^{(-1)^{i+1}}$, *then* $[f] = ([f] \cap [X_0])\, G^*$.

PROOF. For α in $[f]$, define $\beta = \alpha((e\alpha)^{-1})^*$. Then $\beta \in [f] \cap [X^0]$, hence $\alpha = \beta(e\,\alpha)^* \in ([f] \cap [X^0])\, G^*$. Thus since $G^* \subseteq [f]$, $[f] = ([f] \cap [X_0])\, G^*$.

It is easy to show that $[f] \cap [X^0]$ is normal in $[f]$ if and only if $|G| = 1$ or 2, and G^* is normal in $[f]$ if and only if $[f] = [XY^{-1}Z]$.

COROLLARY II. *If* f *in* F *is symmetric and* $[f] \subseteq [XY^{-1}Z]$, *then* G *is abelian.*

PROOF. By Theorem 2.3, $\pi \in [f]$ for every G. Thus for x, z in G, $x^{-1}z^{-1} = x\pi(e\pi)^{-1}z\pi = (xe^{-1}z)\,\pi = (xz)\,\pi = z^{-1}x^{-1}$.

COROLLARY III. $[(XY^{-1})^n X] = [X]$ *if and only if* G *satisfies one of the following.* (a) $g^n = e$ *for all* g *in* G. (b) $g^{n+1} = e$ *for all* g *in* G. (c) $|G| = 3$.

PROOF. Let $x, y \in G$ and $\alpha \in [X]$. If G satisfies (a), then $((xy^{-1})^n x)\,\alpha = x\,\alpha = (x\,\alpha(y\,\alpha)^{-1})^n x\,\alpha$. If G satisfies (b), then $((xy^{-1})^n x)\,\alpha = ((xy^{-1})^{-1}x)\,\alpha = y\,\alpha = (x\,\alpha(y\alpha)^{-1})^{-1}x\,\alpha = (x\alpha(y\alpha)^{-1})^n x\alpha$. If $|G| = 3$, then $|[(XY^{-1})^n X]| \supseteq |[XY^{-1}Z]| = |[XY]|\,|G^*| = 2\cdot 3 = |[X]|$. Conversely suppose that $[(XY^{-1})^n X] = [X]$, then $[X^0] = [(XY^{-1})^n X] \cap [X^0] \subseteq [X^{n+1}]$ by Lemma 2.7. Thus by Theorem 3.4 G satisfies (a), (b) or (c).

COROLLARY IV. *The following are equivalent.* (1) $[XY^{-1}X] = [X]$. (2) $[X] = [X^{-1}]$. (3) $x^2 = e$ *for all* x *in* G. (4) $[X^0] \subseteq [X^{n+1}]$ *for all integers* n. (5) $[(XY^{-1})^n X] = [X]$ *for all integers* n.

PROOF. $[XY^{-1}Z] \subseteq [X^{-1}]$ by Lemma 2.7. Thus (1) implies (2). By Theorem 3.1, (2) implies (3). If G satisfies (3), then G satisfies (a) or (b) of Theorem 3.4, hence (3) implies (4). By Theorem 3.4 and Corollary III, (4) implies (5). Clearly (5) implies (1).

LEMMA 5.1. *If* $g^{n+1} = e$ *for all* g *in* G, *then* $[(XY^{-1})^n Z] = [XY^{-1}Z]$.

PROOF. By Corollary I of Theorem 5.1, $[(XY^{-1})^n Z] = ([(XY^{-1})^n Z] \cap [X^0])\, G^*$. Consider α in $[(XY^{-1})^n Z] \cap [X^0]$ and x, z

in G. $(xz)\alpha = (x^{-n}z)\,\alpha = ((ex^{-1})^n z)\,\alpha = (e\alpha(x\alpha)^{-1})^n z\,\alpha = (x\alpha)^{-n}z\,\alpha = x\alpha\, z\alpha$. Thus $[(XY^{-1})^n Z] = [XY]G^* = [XY^{-1}Z]$.

COROLLARY. $[(XY^{-1})^n Z] = [X]$ *if and only if* G *satisfies one cf the following.* (a) $x^n = e$ *for all* x *in* G. (b) $|G| = 2$ *or* 3 *or* G *is the the four group.*

PROOF. If G satisfies (b), then by counting the elements $[(XY^{-1})^n Z] \supseteq [XY^{-1}Z] = [X]$. If G satisfies (a), then$((xy^{-1})^n z)\alpha = z\alpha = (x\alpha(y\alpha)^{-1})^n z\alpha$ for all x, y, z in G and all α in $[X]$. Thus $[(XY^{-1\,n}Z] = [X]$. Conversely suppose that $[X] = [\,(XY^{-1})^n Z] \subseteq [(XY^{-1})^n X]$. Then by Corollary III of Theorem 5.1, G satisfies (a) or (b) or $g^{n+1} = e$ for all g in G. Suppose that $g^{n+1} = e$ for all g in G. Then $[X] = [(XY^{-1})^n Z] = [XY^{-1}Z] \subseteq [XY^{-1}X]$. Thus by Corollary IV of Theorem 5.1, $g^2 = e$ for all g in G. Also $[X^0] = [X^0] \cap [X] = [X^0] \cap [XY^{-1}Z] = [XY]$. Thus clearly G is a finite abelian group of order 2^m. $(2^m - 1)\,! = |[X^0]| = |[XY]| = (2^m - 1)\,(2^m - 2)\ldots(2^m - 2^{m-1})$. Therefore $m = 1$ or 2.

THEOREM 5.2. $[\Pi X_i^{e_i}] = ([\Pi X_i^{e_i}] \cap [X^0])\,\{g^* : g \in Z(G)$ *and* $g^{\Sigma e_i} = g\}$ *if and only if* $e\,\alpha$ *is in* $Z(G)$ *for all* α *in* $[\Pi X_i^{e_i}]$.

PROOF. Let $A = [\Pi X_i^{e_i}] \cap [X^0]$ and $B = \{g^* \cdot g \in Z(G)$ and $g^{\Sigma e_i} = g\}$. Then $B \subseteq [\Pi X_i^{e_i}]$. For choose g^* in B and $\{y_i\}_{i=1}^n$ in G so that $y_i = y_j$ if $X_i = X_j$. Then $\Pi\,(y_i\, g^*)^{e_i} = \Pi\,(y_i\, g)^{e_i} = (\Pi\, y_i^{e_i})\, g^{\Sigma e_i} = (\Pi\, y_i^{e_i})\, g^*$. Therefore, in any case $[\Pi\, X_i^{e_i}] \supseteq AB$. Now suppose that $e\,\alpha$ is in $Z(G)$ for all α in $[\Pi\, X_i^{e_i}]$. Pick α in $[\Pi X_i^{e_i}]$ and let $e\,\alpha = g$. Then by Lemma 2.3 $g^{\Sigma e_i} = g$. Thus $g^* \in B$ and $\alpha = (\alpha\,(g^{-1})^*)\, g^* \in AB$. Conversely suppose that $[\Pi\, X_i^{e_i}] = AB$, and let $\gamma \in [\Pi\, X_i^{e_i}]$. Then $\gamma = \alpha\, g^*$, where $\alpha \in A$ and $g^* \in B$. Thus $e\,\gamma = e\,\alpha\, g^* = eg^* = g \in Z(G)$.

COROLLARY I. $[\Pi\, X_i^{e_i}] \supset ([\Pi\, X_i^{e_i}] \cap [X^0])\,\{g^* : g \in Z(G)$ *and* $g^{\Sigma e_i} = g\}$.

COROLLARY II. *If* G *is abelian then* $[\Pi\, X_i^{e_i}] = ([\Pi\, X_i^{e_i}] \cap [X^0])\,\{g^* : g \in G$ *and* $g^{\Sigma e_i} = g\}$.

We make use of Theorem 5.2 and the previous results to obtain the structure of the eight groups of the form $[X^{e_1}\, Y^{e_2}\, Z^{e_3}]$.

$[XYZ] = [XY]\ \{g^*: g \in Z(G) \text{ and } g^2 = e\}$. For let $g \in G$ and $\alpha \in [XYZ]$. Then $e\,\alpha\, g\, \alpha\, e\, \alpha = (ege)\,\alpha = (gee)\,\alpha = g\alpha e\alpha e\alpha$. Thus $e\alpha g\alpha = g\,\alpha\, e\,\alpha$ for all g in G. Hence $e\,\alpha \in Z(G)$ and by Theorem 5.2 $[XYZ] = ([XYZ] \cap [X^0])\ \{g^*: g \in Z(G) \text{ and } g^3 = g\}$. But by Lemma 2.7. $[XY] \subseteq [XYZ] \cap [X^0] \subseteq [XY]$. Moreover $g\,\alpha\, g^{-1}\,\alpha\, e\,\alpha = (gg^{-1}\,e)\,\alpha = e\,\alpha$. Thus $g^{-1}\,\alpha = (g\,\alpha)^{-1}$ for all g in G. Therefore $[XYZ] \subseteq [X^{-1}]$. By Lemma 2.3, $[X^{-1}YZ^{-1}]$, $[XY^{-1}\,Z^{-1}]$ and $[X^{-1}\,Y^{-1}Z]$ are all contained in $[X^{-1}]$. By Lemma 2.4, $[XYZ] = [X^{-1}\,YZ^{-1}] = [XY^{-1}\,Z^{-1}] = [X^{-1}\,Y^{-1}Z] \subset [XY^{-1}\,Z]$.

$[XYZ^{-1}] = [X^{-1}YZ] = [XY]\,Z(G)^*$. For consider α in $[XYZ^{-1}]$ and pick z in G such that $z\,\alpha = e$. Then $x\,\alpha\, e\,\alpha = (xez^{-1})\alpha = (exz^{-1})\,\alpha = e\,\alpha\, x\,\alpha$ for all x in G. Thus $e\,\alpha \in Z(G)$ and by Theorem 5.2, $[XYZ^{-1}] = ([XYZ^{-1}] \cap [X^0])\ \{g^*: g\ Z(G) \text{ and } g = g\}$. But by Lemma 2.7, $[XYZ^{-1}] \cap [X^0] = [XY]$. The proof that $[X^{-1}\,YZ] = [XY]Z(G)^*$ is entirely similar.

$[X^{-1}\,Y^{-1}\,Z^{-1}] = [XY]\ \{g^*: g \in Z(G) \text{ and } g^4 = e\}$. For consider α in $[X^{-1}\,Y^{-1}\,Z^{-1}]$ and z in G such that $z\alpha = e$. $(e\,\alpha)^{-1}\,(x\,\alpha)^{-1} = (e^{-1}\,x^{-1}\,z^{-1})\,\alpha = (x^{-1}\,e^{-1}\,z^{-1})\,\alpha = (x\,\alpha^{-1})\,(e\,\alpha)^{-1}$ for all x in G. Thus $e\,\alpha \in Z(G)$ and $[X^{-1}\,Y^{-1}\,Z^{-1}] = ([X^{-1}\,Y^{-1}\,Z^{-1}] \cap [X^0])\ \{g^*: g \in Z(G) \text{ and } g^{-3} = g\}$. By Lemma 2.7, $[X^{-1}\,Y^{-1}\,Z^{-1}] \cap [X^0] \subset [X^{-1}Y^{-1}] \cap [X^0] \subseteq [X^{-1}]$. Thus by Lemma 2.4 $[X^{-1}Y^{-1}] \cap [X^0] = [XY]$.

Therefore

$$\begin{aligned}[XY^{-1}\,Z] &\supseteq [XYZ^{-1}] = [X^{-1}\,YZ] \supseteq [X^{-1}\,Y^{-1}Z^{-1}] \supseteq [XYZ] \\ &= [X^{-1}\,YZ^{-1}] = [X^{-1}\,Y^{-1}\,Z] = [XY^{-1}\,Z^{-1}] \\ &\supseteq [XY].\end{aligned}$$

$[XY^{-1}\,Z] = [XYZ^{-1}]$ if and only if G is abelian. $[XYZ^{-1}] = [X^{-1}\,Y^{-1}\,Z^{-1}]$ if and only if $g^4 = g$ for all g in $Z(G)$. $[X^{-1}Y^{-1}\,Z^{-1}] = [XYZ]$ if and only if $Z(G)$ contains no elements of order 4. $[XYZ] = [XY]$ if and only if $Z(G)$ contains no elements of order 2.

$[XY] = [XY^{-1}] = [X^{-1}\,Y] \subseteq [X^{-1}\,Y^{-1}] = [XY]\ \{g^*: g \in Z(G) \text{ and } g^3 = e\}$. For consider α in $[X^{-1}\,Y^{-1}]$ and x in G. Then

$(x\alpha)^{-1}(e\alpha)^{-1} = (x^{-1}e^{-1})\alpha = (e^{-1}x^{-1})\alpha = (e\alpha)^{-1}(x\alpha)^{-1}$. Thus $e\alpha \in Z(G)$, and $[X^{-1}Y^{-1}] = ([X^{-1}Y^{-1}] \cap [X^0])\{g^* : g \in Z(G)$ and $g^{-2} = g\} = [XY]\{g^* : g \in Z(G)$ and $g^3 = e\}$. By Lemma 2.3, $[X^{-1}Y] \subseteq [X^0]$. Thus by Lemma 2.7, $[X^{-1}Y] = [X^{-1}Y] \cap [X^0] \subseteq [X^{-1}]$. Similarly $[XY^{-1}] \subseteq [X^{-1}]$. Thus by Lemma 2.4, $[XY] = [X^{-1}Y] = [XY^{-1}]$. In particular $[X^{e_1}X^{e_2}] \cap [X^0] = [XY]$ for $e_i = \pm 1$.

THEOREM 5.3. $[X^m Y^n] \cap [X^0] = [XY]$ *for all G if and only if* $|m| = |n| = 1$.

PROOF. Suppose that $[X^m Y^n] \cap [X^0] = [XY]$ for all G. Assume (by way of contradiction) that $|n| > 1$. Case I. $(m,n) = d > 1$. Let G be an abelian group such that $g^d = e$ for all g in G. If $\alpha \in [X^0]$ and $x, y \in G$, then $(x^m y^n)\alpha = e\alpha = (x\alpha)^m (y\alpha)^n$. Thus $[X^m Y^n] \cap [X^0] = [X^0]$, but clearly for large G, $[X^0] \neq [XY]$. Case II. $(m, n) = 1$. Let $G = \langle x_1 \rangle \otimes \langle x_2 \rangle \otimes \langle x_3 \rangle$, where the x_i are of order n. Consider α in $[X]$ and x, y in G. Then $(x^m y^n)\alpha = x^m\alpha$ and $(x\alpha)^m = (x\alpha)^m (y\alpha)^n$. Thus $[X^m Y^n] = [X^m]$. Each x in G has a unique representation $x = (x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$, where $0 \leqslant n_i \leqslant n$. Define $x\alpha = x$ if at least two of the n_i are zero, and define $x\alpha = (x_1^{n_3}, x_2^{n_1}, x_3^{n_2})$ if at least two of the n_i are non-zero. Then $\alpha \in [X^0]$. $((x_1, 0, 0)(0, x_2, 0))\alpha = (x_1, x_2, x_3^0)\alpha = (x_1^0, x_2, x_3) \neq (x_1, x_2, x_3^0) = (x_1, 0, 0)\alpha\,(0, x_2, 0)\alpha$. Thus $\alpha \notin [XY]$. If $x_i^{n_i m} = e$, then $n \mid n_i m$, hence $n \mid n_i$ and $n_i = 0$. It follows that $\alpha \in [X^m] \cap [X^0] = [X^m Y^n] \cap [X^0] = [XY]$, a contradiction.

COROLLARY. *If* $n \neq 1$, *then there exists an abelian group A of order* n^3 *so that* $[X^m Y^n]_A \cap [X^0]_A \neq [XY]_A$.

Thus the quantifier " all groups G " can be replaced by " all finite abelian groups".

THEOREM 5.4. *The following statements are equivalent.* (a) *G is abelian.* (b) $[X_1^{e_1} \ldots X_n^{e_n}] = [X_{\alpha_1}^{e_{\alpha_1}} \ldots X_{\alpha_n}^{e_{\alpha_n}}]$ *for all choice of* n, X_i, e_i *and all permutations* $\alpha_1, \ldots, \alpha_n$. (c) $[XYX^{-1}Y^{-1}] \supseteq [X^0]$. (d) $[XYZ^{-1}] = [XY^{-1}Z]$. (e) $[\Pi X_i^{e_i}] \supset [XY^{-1}Z]$ *provided* $\Sigma e_i = 1$. (f) $[f] = [X]$ *for all f in IC(F), where C(F) is the commutator subgroup of F.* (g) $[XYX^{-1}] = [X]$.

PROOF. (a), (b), (c) and (d) are equivalent by Theorem 2.2, Lemma 2.3 and the remarks after Theorem 5.2. If G is abelian and $\Sigma e_i = 1$ then $[\Pi X_i^{e_i}] \supseteq G^*$. Thus $[\Pi X_i^{e_i}] \supseteq [XY]G^* = [XY^{-1}Z]$, and (a) implies (e). Since $[XYZ^{-1}] \subseteq [XY^{-1}Z]$, (e) implies (d). Clearly (b) implies (f). $YY^{-1}XYX^{-1} \in IC(F)$. Thus if (f) is true, then $[X] = [YY^{-1}XYX^{-1}] = [XYX^{-1}]$. Hence (f) implies (g). Finally suppose that (g) is true and consider a, b in G. Then $\alpha\,(e\,a) \in [X] = [XYX^{-1}]$. If $b = e$ or a, then $ab = ba$. If $b \neq e$ or a then $a = e\,\alpha = (beb^{-1})\,\alpha = b\,\alpha\,e\,\alpha\,(b\,\alpha)^{-1} = bab^{-1}$. Thus G is abelian.

THEOREM 5.5. *If* $[XYX^{-1}Y^{-1}] \supseteq [X^0]$ *and* $[XY^{-1}Z]$ *is a normal splitting extension of a cyclic group* T *by* $[XY]$, *then* $G \simeq T$.

PROOF. By Theorem 5.4, G is abelian. $[XY^{-1}Z] = [XY]T$, $[XY] \cap T = \theta$, T is normal and $T = \langle \alpha t^* \rangle$, where $\alpha \in [XY]$ and $t \in G$. For each $e \neq g$ in G, $g^* = \beta\,(\alpha t^*)^n = \beta\,\alpha^n\,((t\,\alpha^{n-1}) \ldots (t\alpha)t)^*$, where $\beta \in [XY]$ and $n > 0$†. Thus $g = eg^* = (t\,\alpha^{n-1}) \ldots (t\,\alpha)t$. For $\beta \in [XY]$, $\beta^{-1}\,\alpha\,\beta(t\,\beta)^* = \beta^{-1}\,\alpha\,t^*\,\beta = (\alpha\,t^*)^n = \alpha^n((t\,\alpha^{n-1}) \ldots (t\,\alpha)t)^*$. Thus $\beta^{-1}\,\alpha\,\beta = \alpha^n$ and $\langle \alpha \rangle$ is normal in $[XY]$. If $\alpha\,\beta = \beta\alpha$ then $\alpha^{n-1} = \theta$ and $t\,\beta = t\,(t\,\alpha^{n-2}) \ldots (t\,\alpha)\,t$. Thus $(t\,\beta\,t^{-1})\,\alpha = ((t\,\alpha^{n-2}) \ldots (t\,\alpha)t)\,\alpha = t\,(t\,\alpha^{n-2}) \ldots t\,\alpha = t\,\beta\,t^{-1}$. In particular $t^{-2} = t\,\pi\,t^{-1} = (t\,\pi\,t^{-1})\,\alpha = t^{-2}\,\alpha$ and $t\,\alpha\,t^{-1} = (t\,\alpha\,t^{-1})\,\alpha = t\,\alpha^2(t\,\alpha)^{-1}$. Thus $t^2 = t^2\,\alpha = (t\,\alpha)^2 = t\,\alpha^2\,t$ and $t = t\,\alpha^2$. Then $g\,\alpha^2 = ((t\,\alpha^{n-1}) \ldots (t\,\alpha)t)\,\alpha^2 = g$. Therefore $\alpha^2 = \theta$ and $\alpha \in Z([XY])$. Each $\beta \in [XY]$ induces an automorphism β' of $\langle \alpha\,t^* \rangle$: $(\alpha\,t^*)\,\beta' = \beta^{-1}\,\alpha\,t^*\,\beta = \beta^{-1}\,\alpha\,\beta(t\,\beta)^* = \alpha(t\,\beta)^*$. $t\,\beta = t$ if and only if $\beta = \theta$. Thus the mapping of β onto β' is an isomorphism of $[X\,Y]$ onto the group of all automorphisms of the cyclic group T. In particular, $[X\,Y]$ is abelian.

If T is infinite then $(\alpha\,t^*)\,\alpha' = \alpha\,t^*$ or $(\alpha\,t^*)^{-1}$. If $(\alpha\,t^*)\alpha' = \alpha t^*$, then $\alpha = \theta$ and $T = \langle t^* \rangle = G^*$. If $t^*\,\alpha = (\alpha\,t^*)^{-1} = (t^{-1})^*\,\alpha^{-1}$, then $t^* = t^*\,\alpha^2 = (t^*)^{-1}$. Therefore $t^2 = e$ and $\{(t\,\alpha^{n-1}) \ldots (t\,\alpha)t : n > 0\}$ is finite. Hence $[XY^{-1}Z]$ is finite, a contradiction.

Finally suppose that T has finite order m. Then $G = \{(t\,\alpha^{n-1}) \ldots \ldots (t\,\alpha)t\colon 0 < n \leqslant m\}$ is finite and $[XY]$ is abelian. It follows that G is cyclic ([4] p. 35), and since

† For negative n a similar argument holds.

$$|[XY]|\,|G^*| = |[XY]|\,|T| = |[XY]|\,m,\ G \simeq T.$$

Corollary I. *G is cyclic if and only if* $[XYX^{-1}Y^{-1}] \supseteq [X^0]$ *and* $[XY^{-1}Z]$ *is a normal splitting extension of a cyclic group by* $[XY]$.

Corollary II. *If* $[XYX^{-1}Y^{-1}] \supseteq [X^0]$, *then the following are equivalent.* (a) *G is a finite cyclic group of order n.* (b) $|[X]| = n$ *and* $[XY]$ *is abelian.* (c) $|[X]| = n!$ *and* $n+1$ *is the least integer greater than* 1 *for which* $[X^{n+1}] = [X]$. (d) $[XY^{-1}Z]$ *is a normal splitting extension of a finite cyclic group of order n by* $[XY]$.

Corollary III. *If* $[XYX^{-1}Y^{-1}] \supset [X^0]$, *then the following are equivalent.* (a) *G is an infinite cyclic group.* (b) $|[XY]| = 2$ *and* $[XY^{-1}Z]$ *is infinite but finitely generated.* (c) $[XY^{-1}Z]$ *is a normal splitting extension of an infinite cyclic group by* $[XY]$.

For if (a) is true, then $[XY] = \langle \pi \rangle$ and $[XY^{-1}Z]$ is generated by a generator of G and π. If (b) is true, then it follows that G is a finitely generated abelian group. It is easy to show that if G is finitely generated and $|[XY]| = 2$, then G is cyclic.

6. By Theorem 3.1., G is not of bounded order if and only if $[X^n] = [X]$ implies that $n = 1$. This result can be generalized to arbitrary elements of F as follows.

Theorem 6.1. *The following statements are equivalent.* (a) *G is not of bounded order.* (b) *If* $f \in F$ *and* $[f] = [X]$, *then* $f \in IC(F)$.

Proof. Suppose that G is not of bounded order, and $[\Pi\, X_i^{e_i}] = [X]$. Then using the notation of Lemma 2.7, $[X^0] = [X^0] \cap [\Pi\, X_i^{e_i}] \subset [X^q]$, where q is any sum or product of a subsequence of the a_i. $q = 0$ or 1. Otherwise by the corollary to Theorem 3.3, G is of bounded order. $[X] = [\Pi\, X_i^{e_i}] \subseteq [X^{\Sigma e_i}] = [X^{\Sigma a_i}]$. Thus by Theorem 3.1, $\Sigma\, a_i = 1$. It follows that one of the a_i equals 1, say a_j, and $a_i = 0$ for all $a_i \neq a_j$. Therefore $\Pi\, X_i^{e_i} \equiv Z_j \bmod C(F)$, where $Z_j \in I$. Conversely, if G is of bounded order, then by Theorem 3.1, $[X^n] = [X]$ for some $n \neq 1$. If $X^n \in IC(F)$, then $X^n = \Pi\, X_i^{e_i}$, where $\Sigma\, e_i = 1$. Thus (b) is false.

From property (f) of Theorem 5.4 and this theorem we have

COROLLARY. *The following statements are equivalent.* (a) *G is abelian and not of bounded order.* (b) *For all f in F, $[f] = [X]$ if and only if $f \in IC(F)$.*

Each $f = \Pi X_i^{e_i}$ in F determines a unique subset $[[\Pi X_i^{e_i}]]$ of G, where

$$[[\Pi X_i^{e_i}]] = \{g \in G : g = \Pi g_i^{e_i}, \text{ where } g_i = g_j \text{ if } X_i = X_j\}.$$

The mapping of f in F onto $[[\Pi X_i^{e_i}]]$ is independent of the particular representation of f by elements of I, but it does depend upon I. If $\alpha \in [f]$, then $[[f]]\alpha = [[f]]$. One of the relations between the mappings $[\ \ ]$ and $[[\ \ ]]$ is given by the following theorem.

THOREM 6.2. *The following statements are equivalent.* (i) *G is not of bounded order and $f \in C(F) \smallsetminus X^0$ implies $[[f]] \neq e$.* (ii) *If $f \in F$ and $[f] = [X]$, then $f \in I$.*

PROOF. Assume first that (ii) is true. By Theorem 3.1, G is not of bounded order. Suppose (by way of contradiction) that $f \in C(F) \smallsetminus X^0$ and $[[f]] = e$. Then f has an irreducible representation $f = \Pi X_i^{e_i}$, and for any $Y \in I \smallsetminus \{X_i\}_{i=1}^n$, $[Y \Pi X_i^{e_i}] = [Y]$, but $Y \Pi X_i^{e_i} \notin I$, a contradiction. Conversely assume that (i) is true, and that $[\Pi X_i^{e_i}] = [X]$. Then by Theorem 6.1, $\Pi X_i^{e_i} = YB$, where $Y \in I$ and $B \in C(F)$. Suppose (by way of contradiction) that $\Pi X_i^{e_i} \notin I$. Then $B = \prod_{i=1}^{k} B_i^{d_i} \in C(F) \smallsetminus X^0$, where $B_i \in I$ and $d_i = \pm 1$. Therefore there exist $b_1, \dots, b_k$ in G such that $\Pi b_i^{d_i} \neq e$. Replace the B_i with the b_i and replace Y by some y in G. Note that Y may equal one of the B_i. Hence the choice of y is not necessarily arbitrary. Suppose that $y \Pi b_i^{d_i}$ does not belong to $\{b_i\}_{i=1}^k$. Let $y\alpha = y$, $b_i \alpha = b_i$ for $i = 1, \dots, k$, and let $(y \Pi b_i^{d_i})\alpha$ be any element in G other than $y, b_1, \dots, b_k, y \Pi b_i^{d_i}$. Then α can be extended to an element of $[X] = [\Pi X_i^{e_i}] = [Y \Pi B_i^{d_i}]$. Therefore $(y \Pi b_i^{d_i})\alpha \neq y \Pi b_i^{d_i} = y\alpha \Pi (b_i \alpha)^{d_i}$, a contradiction. Thus $y \Pi b_i^{d_i} = b_j$ for some $1 \leqslant j \leqslant k$, and $y \neq b_j$. Choose z in G with order $> k + 1$. Let $y\beta = z$, $b_j \beta = z^{k+1}$, and let β map the other distinct b_i into distinct powers z^s of z, where $1 < s \leqslant k$. Then β can be extended

to an element of $[X]$. Since $\Pi\, b_i^{d_i}$ is the product of commutators, $\Pi (b_i \beta)^{d_i} = e$. Therefore $z = y\beta = y\beta \Pi (b_i \beta)^{d_i} = (y \Pi\, b_i^{d_i})\, \beta = z^{k+1}$, a contradiction.

COROLLARY. *If G is a free group with rank > 1, then $[f] = [X]$ implies $f \in I$ for every f in F.*

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A POINCARE PROBLEM*

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INTRODUCTION. In this paper a boundary value problem connected with the study of neutron flux in a reactor is investigated. Consider an infinite reactor in which the plane segments $y = 0, -1 \leqslant x \leqslant 1$, $x = 0, -1 \leqslant y \leqslant 1$ are occupied by control rods (assumed to be of zero thickness). In the steady state the density $\Phi(x, y)$ of thermal neutrons satisfies the following equations:

$\nabla^2 \Phi(x, y) = \Phi(x, y) - 1$ for all (x, y) not belonging to the arc S consisting of the line segments

$$\left.\begin{aligned} &y = 0, -1 \leqslant x \leqslant 1; \\ &x = 0, -1 \leqslant y \leqslant 1; \\ &\Phi(x, y) = 0 \text{ on } S; \\ &\Phi(x, y) \to 1 \text{ as } \sqrt{(x^2 + y^2)} \to \infty. \end{aligned}\right\} \quad \text{(A)}$$

This is a first approximation to the two-group theory when the fast neutron flux is taken to be constant.

It is shown in Theorem 5 below that problem (A) has a unique solution $\Phi(x, y)$ which can be obtained by solving a non-homogeneous Fredholm equation of the second kind, for which the well-known formulas of Fredholm are applicable. It is also proved that the solution Φ is continuous (in x and y) in the entire x–y plane; and that its gradient is continuous outside S and has a jump discontinuity at points of S, in a direction normal to S. This discontinuity represents the rate of absorption of neutrons by the control

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rods, and is shown (Theorem 5) to satisfy the following conditions: Regarded as a function $\theta'(s)$ of the distance s of the point on S, along the arc S, from some fixed point of S, the rate of absorption is continuous in s except at the extremities of S, namely $x = 0$, $y = \pm 1$, and $y = 0$, $x = +1$, and at the centre $x = y = 0$. At the former points $\theta'(s)$ is of order not higher than $1/\sqrt{(1 \mp y)}$, $1/\sqrt{(1 \mp x)}$ respectively, while at $x = y = 0$, $\theta'(s)$ is of order $\leqslant \log \sqrt{(x^2 + y^2)}$.

THEOREM 1. *The problem* (A) *cannot have two distinct solutions.*

PROOF. This is a direct consequence of a result of Hopf [1, Satz 2], namely that no solution of the homogeneous equation $\nabla^2\phi = \phi$ can have a positive maximum or a negative minimum in an open domain T in which it satisfies the differential equation. If Φ_1, Φ_2 were two distinct solutions of (A), then $\Phi_1 - \Phi_2 = \phi$ would satisfy the equation $\nabla^2\phi = \phi$ in the open domain T formed by removing the arc S from the x-y plane. Also $\Phi_1 - \Phi_2 = 0$ on S and $\to 0$ at infinity, so that if $\Phi_1 - \Phi_2$ differed from zero at any point, then it would attain either a positive maximum or a negative minimum at some finite point of T, the complement of S, contrary to the above mentioned result. Hence

$$\Phi_1 - \Phi_2 \equiv 0.$$

In what follows, the problem (A) is investigated by considering the following associated boundary problems:

$$\left.\begin{aligned}
&\nabla^2 \phi(x, y) = \phi(x, y) \text{ for } 0 < y \leqslant x \text{ as well as} \\
&\qquad\qquad\qquad\qquad \text{for } y = 0, x > 1; \\
&\phi(x, 0) = -1 \text{ for } 0 \leqslant x \leqslant 1; \\
&\frac{\partial \phi}{\partial y} = 0 \text{ for } y = 0, x > 1; \\
&\frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x} = 0 \text{ for } x = y > 0; \\
&\phi(x, y) \to 0 \text{ as } \sqrt{(x^2 + y^2)} \to \infty.
\end{aligned}\right\} \qquad (A_1)$$

$$\left.\begin{aligned} &\nabla^2 \phi(x, y) = \phi(x, y) \text{ for } 0 < y \leqslant x \text{ as well as} \\ &\qquad\qquad\qquad\qquad \text{for } y = 0, x > 1; \\ &\phi(x, 0) = 0 \text{ for } 0 \leqslant x \leqslant 1; \\ &\frac{\partial \phi}{\partial y} = 0 \text{ for } y = 0, x > 1; \\ &\frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x} = 0 \text{ for } x = y > 0; \\ &\phi(x, y) \to 0 \text{ as } (x^2 + y^2) \to \infty. \end{aligned}\right\} \quad (A_2)$$

Suppose that $\phi(x, y)$ is a solution of problem (A_1) which is continuous in $0 \leqslant y \leqslant x$. Then it can be easily verified that the function $\Phi(x, y)$ defined below is a (continuous) solution of problem (A) :

$$\Phi(x, y) = \begin{cases} \phi(x, y) + 1 \text{ for } 0 \leqslant y \leqslant x, \\ \phi(y, x) + 1 \text{ for } 0 \leqslant x \leqslant y; \end{cases}$$

$$\Phi(-x, y) = \Phi(x, y) \text{ for } y \geqslant 0 \text{; and}$$

$$\Phi(x, -y) = \Phi(x, y).$$

Similarly if $\phi(x, y)$ is a solution of problem (A_2) and is continuous in $0 \leqslant y \leqslant x$, then the function $\Phi(x, y)$ defined above satisfies the homogeneous equation

$$\nabla^2 \Phi(x, y) = \Phi(x, y)$$

at all points (x, y) not belonging to S, vanishes on S, and $\to 0$ at infinity. Now as pointed out under Theorem 1, such a function $\Phi(x, y) \equiv 0$. Hence problem (A_2) has only the trivial solution $\phi \equiv 0$.

In Theorems 2–5 below a non-homogeneous Fredholm equation (E) (of the second kind) is constructed satisfying the following:

the kernal $M(\beta, \beta')$ of (E) is continuous in both variables; (1)

every solution of (E) defines a continuous solution ϕ of (A_1); (2)

every non-trivial solution of the homogeneous Fredholm equation (E′) corresponding to (E) defines a non-trivial continuous solution of (A_2). (3)

Since (A_2), as shown above, has no non-trivial solution, therefore by (3) (E′) also has no non-trivial solution. Hence by the well-known theory of Fredholm equations, (E) has a unique solution which can be obtained by Fredholm's formula, and which, by (2) determines the unique solution of (A_1), and hence of (A).

The proof of these results makes use of the function K defined below. Using the results given in ([4], p. 100) it might be verified that K is a Neumann function for the problem (A_1). Let

$$K(x, y; \xi, \eta) = \frac{\pi}{2i} \operatorname{Im} \{H_0'(i\rho_1) + H_0'(i\rho_2) + \dots + H_0'(i\rho_8)\},$$

where

$$\rho_1^2 = (x - \xi)^2 + (y - \eta)^2,$$

$$\rho_2^2 = (x - \eta)^2 + (y - \xi)^2,$$

$$\rho_3^2 = (x - \xi)^2 + (y + \eta)^2,$$

$$\rho_4^2 = (x + \eta) + (y - \xi)^2,$$

$$\rho_5^2 = (x + \xi)^2 + (y - \eta)^2,$$

$$\rho_6^2 = (x - \eta)^2 + (y + \xi)^2,$$

$$\rho_7^2 = (x + \xi)^2 + (y + \eta)^2,$$

$$\rho_8^2 = (x + \eta)^2 + (y + \xi)^2.$$

THEOREM 2. *Suppose that the integral equation*

$$\int_0^1 \theta(\xi) K(x, 0; \xi, 0)\, d\xi = -2\pi \tag{a}$$

has a solution $\theta(\xi)$ satisfying the following conditions:

$|\theta(\xi)|$ *is integrable in* $(0, 1)$, (4)

$\theta(\xi)$ *is continuous in* $0 < \xi < 1$, (5)

$\theta(\xi)\sqrt{(1 - \xi)}$ *is continuous at* $\xi = 1$, (6)

$\dfrac{\theta(\xi)}{\log \xi}$ *is bounded in* $0 \leqslant \xi \leqslant 1/2$. (7)

Then the function $\phi(x, y)$ defined by

$$2\pi\,\phi(x, y) = \int_0^1 \theta(\xi)\,K(x, y; \xi, 0)\,d\xi \tag{b}$$

is a continuous solution of problem (A_1), *and*

$$\lim_{y \to +0} \frac{\partial \phi}{\partial y}(x, y) = -\theta(x) \text{ for } 0 < x < 1.$$

Similarly if $\bar{\theta}(\xi)$ *is a non-trivial solution of the integral equation*

$$\int_0^1 \bar{\theta}(\xi)\,K(x, 0; \xi, 0)\,d\xi = 0, \tag{a'}$$

and satisfies conditions (4) – (7), *then the function* $\bar{\phi}(x, y)$ *defined by*

$$2\pi\,\bar{\phi}(x, y) = \int_0^1 \bar{\theta}(\xi)\,K(x, y; \xi, 0)\,d\xi \tag{b'}$$

is a continuous non-trivial solution of (A_2), *and*

$$\lim_{y \to 0} \frac{\partial \bar{\phi}}{\partial y} = -\bar{\theta}(x) \text{ for } 0 < x < 1.$$

In what follows, the class of functions θ which satisfy conditions (4) – (7) will be denoted by (γ).

PROOF : The only singularities of the function $H_0^1(i\rho)$ are $\rho = 0$ and $\rho = \infty$; also ([4], p. 317)

$$\lim_{\rho \to 0} H_0^1(i\rho) = \frac{2i}{\pi} \log \frac{\bar{\gamma}\, i \rho}{2} + \text{constant};$$

and ([4], p. 100)

$$H_0^1(i\rho) = O(e^{-\rho}) \text{ as } \rho \to \infty.$$

Hence the only discontinuities of $K(x, y; \xi, \eta)$ are the points $\rho_i = 0$, $i = 1, 2, \ldots, 8$. In the region under consideration, namely $0 \leqslant y \leqslant x$, for $\eta = 0$, these reduce to the single point $x = \xi$, $y = 0$, at which $K = O(\log \sqrt{\{(x-\xi)^2 + y^2\}})$. Hence conditions (4) – (7) ensure that the integral

$$\phi(x, y) = \frac{1}{2\pi} \int_0^1 K(x, y; \xi, 0)\,\theta(\xi)\,d\xi$$

exists and is a continuous function of x and y in $0 \leqslant y \leqslant x$.

Now let $\sqrt{(x^2+y^2)} \to \infty$. For $0 \leqslant \xi \leqslant 1$, each $\rho_i = O(\sqrt{(x^2+y^2)}) \to \infty$, and hence

$$K(x, y; \xi, 0) = O(e^{-\sqrt{(x^2+y^2)}}).$$

Therefore $\phi(x, y)$ is also $= O(e^{-\sqrt{(x^2+y^2)}}) \to 0$.

It can be verified that for given (x, y) such that either $y > 0$, or $y = 0, x > 1$,

$$2\pi \frac{\partial^n \phi}{\partial x^n}(x, y) = \int_0^1 \theta(\xi)\, d\xi . \frac{\partial^n}{\partial x^n} K(x, y; \xi, 0),\ n \geqslant 1,$$

and similarly for $\dfrac{\partial^n \phi}{\partial y^n}$, since $\dfrac{\partial^n}{\partial x^n} K(x, y; \xi, 0)$ is continuous in ξ in $0 \leqslant \xi \leqslant 1$ (where each ρ_i has a lower bound > 0). Therefore the equations

$$\nabla^2 \phi(x, y) = \phi(x, y) \text{ for } y > 0, \text{ and for } y = 0,\ x > 1;$$

$$\frac{\partial \phi}{\partial y}(x, y) = 0 \text{ for } y = 0, x > 1;$$

and

$$\frac{\partial \phi}{\partial x}(x, y) = \frac{\partial \phi}{\partial y}(x, y) \text{ for } x = y > 0$$

follow from the corresponding equations for K. (The first of these, namely $K_{xx} + K_{yy} = K$, follows from the similar equation satisfied by $H_0^1(i\rho)$; the latter two, from the symmetry of K about the lines $y = 0$ and $x = y$).

The equation

$$\lim_{y \to 0} \frac{\partial \phi}{\partial y}(x, y) = -\theta(x) \text{ for } 0 < x < 1$$

follows by taking the limit as $y \to +0$ of both sides of

$$2\pi \frac{\partial \phi}{\partial y}(x, y) = \int_0^1 \theta(\xi) . \frac{\partial K}{\partial y}(x, y; \xi, 0)\, d\xi, \tag{b}$$

and observing that

$$\lim_{y \to 0} \frac{\partial K}{\partial y} (x, y; \xi, 0) = 0 \text{ for } x \neq \xi,$$

and that for any $\epsilon > 0$,

$$\lim_{y \to 0} \int_{-\epsilon}^{+\epsilon} \frac{\partial K}{\partial y} (x, y; \xi, 0)\, d\xi = -2\pi.$$

(This last result follows from Green's theorem, since $K(x, y; \xi, 0) = 2 \log \sqrt{\{(x-\epsilon)^2 + y^2\}}$ + a differentiable function.)

An exactly similar proof holds for the second part of Theorem 2, i.e. to show that $\bar{\phi}$ is a continuous solution of (A_2), and that

$$\lim_{y \to 0} \frac{\partial \bar{\phi}}{\partial y} (x, y) = -\bar{\theta}(x) \text{ for } 0 < x < 1,$$

whence it follows that if $\bar{\theta}$ is not $\equiv 0$, then $\bar{\phi}$ is also not $\equiv 0$. This completes the proof of Theorem 2.

The proofs of Theorems 3 and 4 make use of the following lemmas and their corrollaries :

LEMMA 1. *The function*

$$\chi(\rho) = \frac{\pi}{2i} \operatorname{Im} \{H'_0(i\rho)\} - \log \rho$$

is continuous in $0 \leqslant \rho \leqslant 1$, *and* $\chi(0) = 0$.

LEMMA 2. $\chi'(\rho) = \dfrac{d}{d\rho} \chi(\rho)$ *exists and is continuous and* $\leqslant 0$ *in* $0 \leqslant \rho \leqslant 1$, *and* $\chi'(0) = 0$.

LEMMA 3. $\chi''(\rho) = \dfrac{d^2}{d\rho^2} \chi(\rho)$ *exists and is continuous in* $0 < \rho \leqslant 1$ *and* $= O(\log \rho)$ *as* $\rho \to 0$, *i.e.*

$$\left| \frac{\chi''(\rho)}{\log \rho} \right| \text{ is bounded as } \rho \to 0.$$

PROOF : Lemma 1 follows from the results of ([4], p. 317). The remaining results follow from the identity

$$H_0^1(i\rho) = -\frac{1}{\pi}\left\{\int_{-i\infty}^{0} + \int_{0}^{-\pi} + \int_{-\pi}^{-\pi+i\infty}\right\}\{e^{\rho\sin\xi}\,d\xi\}$$

$$= -\frac{1}{\pi}\left\{2i\int_0^\infty e^{-i\rho\sinh\eta}\,d\eta + \int_0^{-\pi} e^{\rho\sin\xi}\,d\xi\right\},$$

whence

$$\operatorname{Im}\{H_0^1(i\rho)\} = -\frac{2i}{\pi}\int_0^\infty \cos(\rho\sinh\eta)\,d\eta = -\frac{2i}{\pi}\int_0^\infty \frac{\cos\lambda}{\sqrt{(\rho^2+\lambda^2)}}\,d\lambda.$$

Hence for $\rho > 0$ we have

$$\chi(\rho) = \int_0^\infty \frac{\cos\lambda}{\sqrt{(\rho^2+\lambda^2)}}\,d\lambda - \log\rho$$

and

$$\chi'(\rho) = -2\rho\int_0^\infty \frac{\sin^2\lambda/2}{(\rho^2+\lambda^2)^{3/2}}\,d\lambda.$$

Hence $\chi'(\rho)$ clearly exists and is continuous and $\leqslant 0$ in $0 < \rho \leqslant 1$. Also in the neighbourhood of $\rho = 0$,

$$\left|\frac{\chi'(\rho)}{\rho\log\rho}\right| \leqslant 2\left|\frac{\int_0^1 \frac{\sin^2\lambda/2\,d\lambda}{(\rho^2+\lambda^2)^{3/2}}}{\log\rho}\right| + 2\left|\frac{\int_1^\infty \frac{\sin^2\lambda/2\,d\lambda}{(\rho^2+\lambda^2)^{3/2}}}{\log\rho}\right|$$

$$\leqslant \frac{1}{2}\left|\frac{\int_0^1 \frac{\lambda^2\,d\lambda}{(\rho^2+\lambda^2)^{3/2}}}{\log\rho}\right| + 2\left|\frac{\int_1^\infty \frac{d\lambda}{\lambda^3}}{\log\rho}\right|,$$

which can be shown to be bounded as $\rho \to 0$. Hence $\lim \chi'(\rho)$ exists and is $= 0$. This proves Lemma 2.

Now

$$\frac{d^2\chi}{d\rho^2} + \frac{1}{\rho}\frac{d\chi}{d\rho} = \frac{\pi}{2i}\cdot \operatorname{Im}\left\{\frac{d^2H_0^1}{d\rho^2} + \frac{1}{\rho}\frac{dH_0^1}{d\rho}\right\}$$

$$= \frac{\pi}{2i} \cdot \operatorname{Im}\{H_0^1(i\rho)\} = \chi(\rho) + \log \rho.$$

It follows that $\chi''(\rho) = \dfrac{d^2\chi}{d\rho^2}$ exists and is continuous in $\rho > 0$. Also dividing both sides by $\log \rho$ and letting $\rho \to 0$, it follows that $\dfrac{\chi''(\rho)}{\log \rho}$ is bounded as $\rho \to 0$ (since all the other terms in the equation are bounded). This proves Lemma 3.

Corollary 1. *The function*

$$\mu(x, \xi) = \frac{1}{2}\left\{ K(x, 0; \xi, 0) - 2 \log |x^4 - \xi^4| \right\}$$

is continuous in x and ξ in the rectangle R given by

$$0 \leqslant x \leqslant 1,\ 0 \leqslant \xi \leqslant 1.$$

Corollary 2. $\dfrac{\partial \mu}{\partial x}(x, \xi)$ *exists and is continuous in* $0 \leqslant x \leqslant 1$, $0 \leqslant \xi \leqslant 1$, *and* $= 0$ *for* $x = 0$.

Corollary 3. $\dfrac{\partial^2 \mu}{\partial x^2}$ *exists and is continuous at all points of the rectangle R other than those on $x = \xi$; for $x \neq \xi$,* $\left|\dfrac{\partial^2 \mu}{\partial x^2}(x, \xi)/\log|x - \xi|\right|$ *is bounded uniformly in R as $x \to \xi$.*

Proof : It can be easily verified that

$$K(x, 0; \xi, 0) = \frac{\pi}{2i} \cdot \operatorname{Im}\{2H_0^1(i|x - \xi|) + 2H_0^1(i|x + \xi|) + \\ + 4H_0^1(i\sqrt{(x^2 + \xi^2)})\}$$

and hence, that

$$\mu(x, \xi) = \chi(|x - \xi|) + \chi(|x + \xi|) + 2\chi(\sqrt{(x^2 + \xi^2)}).$$

Corollary 1 follows directly from Lemma 1 ; and Corollaries 2 and 3 can be deduced from Lemmas 2 and 3 by differentiating the above expression for $\mu(x, \xi)$, with respect to x.

The following result is immediate from Corollary 3.

COROLLARY 4. *If $\Delta x > 0$ and $(\bar{x}, \xi)$ and $(\bar{x} + \Delta x, \xi)$ both belong to the rectangle R, then*

$$\left| \frac{\left.\frac{\partial}{\partial x}\mu(x, \xi)\right]_{x=\bar{x}+\Delta\bar{x}} - \left.\frac{\partial}{\partial x}\mu(x, \xi)\right]_{x=\bar{x}}}{\Delta x} \right| \leqslant K_1 \,|\log \Delta x\,| + K_2 + \\ + K_3 \,|\log \sqrt{(\bar{x}^2 + \xi^2)}\,|,$$

(where K_1, K_2, K_3 are constants).

LEMMA 4. *If $\theta(\xi) \in (\gamma)$, then $\int\limits_0^1 \theta(\xi) \frac{\partial^2}{\partial x^2} \mu(x, \xi)\, d\xi$ exists and is continuous in x in $0 \leqslant x \leqslant 1$, and*

$$= \frac{d}{dx}\left\{\int\limits_0^1 \theta(\xi) \frac{\partial}{\partial x} \mu(x, \xi)\, d\xi\right\}$$

$$= \frac{d^2}{dx^2}\left\{\int\limits_0^1 \theta(\xi)\, \mu(x, \xi)\, d\xi\right\}.$$

(The latter integrals are also continuous in $0 \leqslant x \leqslant 1$.)

Also $\frac{1}{x}\int\limits_0^1 \theta(\xi) \frac{\partial}{\partial x} \mu(x, \xi)\, d\xi$ is continuous in $0 \leqslant x \leqslant 1$.

These results can be deduced from Corollaries 1-4.

THEOREM 3. *Consider the integral equation*

$$\theta(\xi) = \frac{2}{\sqrt{(1 - \xi^4)}}\left\{\xi k + \int\limits_0^1 L(\xi, \xi')\, \theta(\xi')\, d\xi'\right\}, \qquad \text{(c)}$$

where

$$k = \frac{1}{\pi \log 2}\int\limits_0^1 \frac{4x}{\sqrt{(1 - x^4)}}\, dx$$

and

$$L(\xi, \xi') = \frac{\xi}{\pi^2 \log 2}\int\limits_0^1 \frac{4x}{\sqrt{(1 - x^4)}}\, \mu(x, \xi')\, dx -$$

$$- \frac{2}{\pi^2} \int_0^1 \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi') \, dx$$

for $0 \leqslant \xi' \leqslant 1$, $0 < \xi \leqslant 1$ *(and undefined for* $\xi = 0$*). Every solution* θ *of equation* (c) *which belongs to* (γ) *is a solution of equation* (a) *also.*

Similarly every solution $\bar{\theta}$ *of the corresponding homogeneous equation*

$$\bar{\theta}(\xi) = \frac{2}{\sqrt{(1-\xi^4)}} \int_0^1 L(\xi, \xi') \, \bar{\theta}(\xi') \, d\xi' \qquad (c')$$

which belongs to (γ) *is a solution of the homogeneous equation* (a').

PROOF. Using the notation of Corollary 1, equation (a) can be written as

$$\int_0^1 \log(|x^4 - \xi^4|) . \, \theta(\xi) \, d\xi = -\pi - \int_0^1 \mu(x, \xi) \, \theta(\xi) \, d\xi = f(x).$$

Writing $x^4 = \frac{1+\cos s}{2}$, $\xi^4 = \frac{1+\cos t}{2}$, and $\frac{\theta(\xi)}{4\xi^3} = u(t)$, the equation becomes

$$\int_0^\pi [\log|\cos s - \cos t| - \log 2] \, u(t) \sin t \, dt \doteq 2f(x) = \bar{f}(s). \qquad (c_1)$$

$\frac{d\bar{f}}{ds}$ is square integrable in $(0, \pi)$. For,

$$\frac{d\bar{f}}{ds} = 2f'(x) . \, \frac{dx}{ds} = \frac{\sin s}{4x^3} \int_0^1 \theta(\xi) \, d\xi \, \frac{\partial}{\partial x} \mu(x, \xi)$$

$$= \frac{1}{4x} \int_0^1 \theta(\xi) \frac{\partial}{\partial x} \mu(x, \xi) \, d\xi \, . \, \frac{\sin s}{\{(1+\cos s)/2\}^{1/2}}$$

which, by Lemma 4, is continuous in x and s in $0 \leqslant x \leqslant 1$, $0 \leqslant s \leqslant \pi$ while x is a continuous function of s in $(0, \pi)$. Hence by the method of ([3], p. 68) it can be shown that the equation (c_1) has the formal solution

$$u(t) \sin t = \frac{-1}{2\pi^2 \log 2} . \int_0^\pi \bar{f}(s) \, ds - \frac{1}{\pi^2} \int_0^\pi \frac{\bar{f}'(s) . \sin s \, ds}{\cos s - \cos t} .$$

In terms of the old variables x and ξ this becomes

$$\frac{\theta(\xi)}{4\xi^3} 2\sqrt{\{\xi^4(1-\xi^4)\}} = \frac{-2}{2\pi^2 \log 2} \int_0^1 \frac{f(x).\ 4x^3}{\sqrt{\{x^4(1-x^4)\}}} dx + \\ + \frac{2}{\pi^2} \int_0^1 \frac{f'(x).\ \sqrt{\{x^4(1-x^4)\}}}{x^4-\xi^4} dx,$$

or equivalently

$$\theta(\xi) = \frac{-2\xi}{\pi^2 \log 2\sqrt{(1-\xi^4)}} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}} dx \times \\ \times \left\{ -\pi - \int_0^1 \mu(x, \xi')\ \theta(\xi)\ d\xi' \right\} + \\ + \frac{4\xi}{\pi^2\sqrt{(1-\xi^4)}} \int_0^1 \frac{x^2\sqrt{(1-x^4)}}{x^4-\xi^4} dx \times \\ \times \left\{ -\int_0^1 \theta(\xi') \frac{\partial}{\partial x} \mu(x, \xi') d\xi' \right\} \tag{d}$$

Hence every solution $\theta(\xi)$ of equation (d) which belongs to (γ) is a solution of (a) also.

Similarly it can be shown that every solution $\bar{\theta}(s)$ of the homogeneous equation

$$\bar{\theta}(\xi) = \frac{-2\xi}{\pi^2 \log 2\sqrt{(1-\xi^4)}} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}} dx \left\{ -\int_0^1 \mu(x, \xi')\ \bar{\theta}(\xi')\ d\xi' \right\} + \\ + \frac{4\xi}{\pi^2\sqrt{(1-\xi^4)}} \int_0^1 \frac{x^2\sqrt{(1-x^4)}}{x^4-\xi^4} dx \times \\ \times \left\{ -\int_0^1 \bar{\theta}(\xi') \frac{\partial}{\partial x} \mu(x, \xi')\ d\xi' \right\} \tag{d'}$$

which belongs to (γ) is a solution of (a′) also.

Hence to prove Theorem 3 it is sufficient to show that in the class (γ), the solutions of equations (c) and (d) are identical, and similarly

the solutions of (c′) and (d′) are identical. This will clearly be true if for $0 < \xi < 1$ and any $\theta \in (\gamma)$,

$$\frac{2\xi}{\pi^2 \log 2 . \sqrt{(1-\xi^4)}} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}}\, dx \int_0^1 \mu(x, \xi')\, \theta(\xi')\, d\xi' -$$

$$- \frac{4\xi}{\pi^2 \sqrt{(1-\xi^4)}} \int_0^1 \frac{x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4}\, dx \int_0^1 \theta(\xi')\, d\xi' \frac{\partial}{\partial x} \mu(x, \xi')$$

$$= \frac{2}{\sqrt{(1-\xi^4)}} \int_0^1 L(\xi, \xi')\, \theta(\xi')\, d\xi'.$$

Using Lemma 4 it can be shown that for $0 < \xi < 1$

$$\frac{2\xi}{\pi^2 \log 2 . \sqrt{(1-\xi^4)}} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}}\, dx \int_0^1 \mu(x, \xi')\, \theta(\xi')\, d\xi'$$

$$= \frac{2}{\sqrt{(1-\xi^4)}} \cdot \frac{\xi}{\pi^2 \log 2} \int_0^1 \theta(\xi')\, d\xi' \int_0^1 \frac{4x}{\sqrt{(1-x^4)}}\, \mu(x, \xi')\, dx.$$

Hence it remains to show that for $0 < \xi < 1$

$$\frac{4}{\pi^2 \sqrt{(1-\xi^4)}} \int_0^1 \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4}\, dx \int_0^1 \theta(\xi') \frac{\partial}{\partial x} \mu(x, \xi')\, d\xi'$$

$$= \frac{4}{\pi^2 \sqrt{(1-\xi^4)}} \int_0^1 \theta(\xi')\, d\xi' \int_0^1 \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx.$$

Let $\epsilon > 0$ be such that $0 < \xi - \epsilon < \xi + \epsilon < 1$. Clearly

$$\left(\int_0^{\xi-\epsilon} + \int_{\xi+\epsilon}^1 \right) \left(\frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4}\, dx . \int_0^1 \theta(\xi') \frac{\partial}{\partial x} \mu(x, \xi')\, d\xi' \right)$$

$$= \int_0^1 \theta(\xi')\, d\xi' . \left(\int_0^{\xi-\epsilon} + \int_{\xi+\epsilon}^1 \right) \left(\frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \right)$$

so that it is sufficient to show that the principal values of

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4}\, dx \int_0^1 \theta(\xi') \frac{\partial}{\partial x} \mu(x, \xi')\, d\xi'$$

and

$$\int_0^1 \theta(\xi')\, d\xi' \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, d\xi' \to 0 \text{ as } \epsilon \to 0.$$

The first result follows from Theorem 1 of ([2] p. 115), by observing that $\dfrac{1}{x^4-\xi^4} = \dfrac{1}{x-\xi} \cdot \dfrac{1}{(x^2+\xi^2)(x+\xi)}$, and that each of the functions $\dfrac{\xi x^2 \sqrt{(1-x^4)}}{(x^2+\xi^2)(x+\xi)}$, $\displaystyle\int_0^1 \theta(\xi') \frac{\partial}{\partial x} \mu(x, \xi')\, d\xi'$ satisfies, for given $\xi > 0$, a Lipschitz condition with exponent $\frac{1}{2}$ in $0 \leqslant x \leqslant 1$. (The last function possesses, by Lemma 4, a continuous derivative in $0 \leqslant x \leqslant 1$.)

Since $|\theta|$ is integrable in $(0, 1)$, therefore the second result will hold provided

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \to 0 \text{ as } \epsilon \to 0,$$

uniformly with respect to ξ' in $0 \leqslant \xi' \leqslant 1$. This again follows by an obvious generalization of Theorem 1 of [2]. For, the integrand is $= \dfrac{f(x, \xi')}{x-\xi}$, where

$$f(x, \xi') = \frac{\xi x^2 \sqrt{(1-x^4)}}{(x^2+\xi^2)(x+\xi)} \frac{\partial}{\partial x} \mu(x, \xi')$$

satisfies, for given $\xi > 0$, a Lipschitz condition with exponent $\frac{1}{2}$ in $0 \leqslant x \leqslant 1$, uniformly with respect to ξ' in $0 \leqslant \xi' \leqslant 1$. Hence as in [2],

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{f(x, \xi')}{x-\xi}\, dx = \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{f(x, \xi') - f(\xi, \xi')}{x-\xi}\, dx + \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{dx}{x-\xi} f(\xi, \xi')$$

$$\leqslant K \int_{\xi-\epsilon}^{\xi+\epsilon} dx.\ \sqrt{(|x-\xi|)} \to 0$$

as $\epsilon \to 0$, uniformly in $0 \leqslant \xi' \leqslant 1$. This proves Theorem 3.

THEOREM 4. (i) *The function* $L(\xi, \xi')$ *defined in Theorem 3 is a continuous function of* ξ *and* ξ' *in* $0 < \xi \leqslant 1$, $0 \leqslant \xi' \leqslant 1$;

(ii) $\dfrac{L(\xi, \xi')}{\log \xi}$ *is bounded in the neighbourhood of* $\xi = 0$, *uniformly relative to* ξ' *in* $0 \leqslant \xi' \leqslant 1$.

REMARK : It may be noted that (ii) is satisfied by any function L which is continuous in ξ and ξ' in $0 \leqslant \xi \leqslant 1, 0 \leqslant \xi' \leqslant 1$. (ii) is also satisfied if in addition to (i), the following condition holds.

(ii)$'$ For any arbitrary function $\rho(\xi)$ of ξ which is continuous and monotonic in $0 \leqslant \xi \leqslant \frac{1}{2}$ and vanishes at $\xi = 0$, $\dfrac{\rho(\xi)}{\log \xi} L(\xi, \xi') \to 0$ uniformly as $\xi \to 0$, for $0 \leqslant \xi' \leqslant 1$.

For, if (ii) is violated, then it is possible to define in $0 \leqslant \xi \leqslant \frac{1}{2}$ a monotonic continuous function $\rho(\xi)$ such that $\rho(0) = 0$, and (ii)$'$ is violated.

PROOF OF THEOREM 4. We find by Lemma 4 (since the function $\dfrac{4x}{\sqrt{(1-x^4)}} \in(\gamma)$), that

$$\frac{\xi}{\pi^2 \log 2} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}} \mu(x, \xi')\, dx = L_1(\xi, \xi')$$

is continuous in ξ and ξ' in $0 \leqslant \xi \leqslant 1$, $0 \leqslant \xi' \leqslant 1$ (and hence satisfies (i) and (ii)). Therefore it is sufficient to prove (i) and (ii) or equivalently, (i), (ii)$'$ for the function

$$L_2(\xi, \xi') = \int_0^1 \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx.$$

Let $\epsilon > 0$.

$$L_2(\xi, \xi') = \left(\int_0^{\max(0, \xi - \epsilon)} + \int_{\min(\xi + \epsilon, 1)}^1 \right) \times \left(\frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \right) +$$

$$+\int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2\sqrt{(1-x^4)}}{x^4-\xi^4}\frac{\partial}{\partial x}\mu(x,\xi')\,dx.$$

For given $\epsilon > 0$ the first term on the right is evidently continuous in ξ and ξ' in $0 \leqslant \xi \leqslant 1$, $0 \leqslant \xi' \leqslant 1$. Hence (i) will follow if the second term on the right $\to 0$ as $\epsilon \to 0$, uniformly in any rectangle of the form $(0 <) \Lambda \leqslant \xi \leqslant 1$, $0 \leqslant \xi' \leqslant 1$. Similarly (ii)$'$ will follow if for $0 < \epsilon \leqslant 1$, $0 \leqslant \xi \leqslant \Lambda (< 1)$, and $0 \leqslant \xi' \leqslant 1$,

$$\frac{\rho(\xi)}{\log \xi}\int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2\sqrt{(1-x^4)}}{x^4-\xi^4}\frac{\partial}{\partial x}\mu(x,\xi')\,dx \to 0$$

uniformly as $\Lambda \to 0$, where $\rho(\xi)$ is any given continuous monotonic function in $0 \leqslant \xi \leqslant \frac{1}{2}$ such that $\rho(0) = 0$.

Let Λ, $\bar{\Lambda}$ be such that $0 < \Lambda < \bar{\Lambda} < 1$.

Case 1. $\Lambda \leqslant \xi \leqslant \bar{\Lambda}$. Since $\epsilon > 0$ is arbitrarily small it can be assumed that $0 \leqslant \xi - \epsilon < \xi + \epsilon \leqslant 1$. As in the proof of Theorem 3, the required result follows by a modification of Theorem 1 of [2]. For,

$$\int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2\sqrt{(1-x^4)}}{x^4-\xi^4}\frac{\partial}{\partial x}\mu(x,\xi')\,dx = \int_{\xi-\epsilon}^{\xi+\epsilon}\frac{f(x,\xi,\xi')}{x-\xi}\,dx,$$

where

$$f(x,\xi,\xi') = \frac{\xi x^2\sqrt{(1-x^4)}}{(x^2+\xi^2)(x+\xi)}\frac{\partial}{\partial x}\mu(x,\xi').$$

It can be verified that $\dfrac{\partial}{\partial x}\left(\dfrac{\xi x^2}{(x^2+\xi^2)(x+\xi)}\right) \leqslant \dfrac{K}{\xi^2}$

for $x \geqslant 0$, $\xi > 0$, and hence is continuous and bounded in $\Lambda \leqslant \xi \leqslant \bar{\Lambda}$. Hence by the same arguments as before it follows that $f(x, \xi, \xi')$ satisfies a Lipschtiz condition with exponent $\frac{1}{2}$ in $0 \leqslant x \leqslant 1$, uniformly with respect to ξ, ξ' in $\bar{\Lambda} \leqslant \xi \leqslant \Lambda$, $0 \leqslant \xi' \leqslant 1$, and therefore, for $\Lambda \leqslant \bar{\xi} \leqslant \Lambda$, $0 \leqslant \xi' \leqslant 1$,

$$\int_{\xi-\epsilon}^{\xi+\epsilon}\frac{f(x,\xi,\xi')}{x-\xi}\,dx \to 0 \text{ uniformly as } \epsilon \to 0.$$

CASE 2. $0 < \xi < \Lambda$. It can be assumed without loss of generality that $\Lambda + \epsilon < 1$. Consider the two possibilities :

CASE B. $0 < \epsilon \leqslant \xi (< \Lambda)$. It can be easily verified that

$$\int\limits_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx = \frac{I_1 - I_2}{2},$$

where

$$I_1 = I_1(\xi, \xi', \epsilon) = \int\limits_{\xi-\epsilon}^{+\epsilon} \frac{x^2\sqrt{(1-x^4)}}{x^2+\xi^2} \frac{\partial}{\partial x} \mu(x, \xi') . \frac{dx}{x-\xi},$$

$$I_2 = I_2(\xi, \xi', \epsilon) = \int\limits_{\xi-\epsilon}^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} . \sqrt{(1-x^4)} . \frac{\partial}{\partial x} \mu(x, \xi') . \frac{dx}{x+\xi} .$$

Now $\left|\frac{1}{x+\xi}\right| \leqslant \frac{1}{\xi}$ for $0 \leqslant \xi - \epsilon \leqslant x \leqslant \xi + \epsilon$; and $\frac{x^2\sqrt{(1-x^4)}}{x^2+\xi^2}$ as well as $\frac{\partial}{\partial x} \mu(x, \xi')$ are uniformly bounded in $\xi - \epsilon \leqslant x \leqslant \xi + \epsilon$, $0 < \xi \leqslant \Lambda$, $0 \leqslant \xi' \leqslant 1$. Hence

$$|I_2| \leqslant \frac{K}{\xi} \int\limits_{\xi-\epsilon}^{\xi+\epsilon} dx = \frac{2K\epsilon}{\xi} \leqslant 2K$$

since $\epsilon \leqslant \xi$ by hypothesis (K being independent of ξ, ξ' and ϵ).

Writing $|x - \xi| = x'$ we have

$$I_1 = \int\limits_0^{\epsilon} \frac{dx'}{x'} \left\{ \frac{(x'+\xi)^2}{(x'+\xi)^2+\xi^2} . \sqrt{(1-(x'+\xi)^4)} . \frac{\partial}{\partial x} \mu(x, \xi')\right]_{x=x'+\xi}$$

$$- \frac{(\xi - x')^2}{(\xi-x')^2+\xi^2} \sqrt{(1-(\xi-x')^4)} . \frac{\partial}{\partial x} \mu(x, \xi')\Big]_{x=\xi-x'} \Bigg\},$$

which can be written as $J_1 + J_2 + J_3$, where

$$J_1 = J_1(\xi, \xi', \epsilon) = \int\limits_0^{\epsilon} \frac{dx'}{x'} \frac{(\xi+x')^2}{(\xi+x')^2+\xi^2} \sqrt{(1-(\xi+x')^4)} \times$$

$$\times \left\{ \frac{\partial}{\partial x} \mu(x, \xi')]_{x=\xi+x'} - \frac{\partial}{\partial x} \mu(x, \xi')]_{x=\xi-x'} \right\},$$

$$J_2(\xi, \xi', \epsilon) = \int_0^\epsilon \frac{dx'}{x'} \sqrt{(1-(\xi+x')^4)} \frac{\partial}{\partial x} \mu(x, \xi')]_{x=\xi-x'} \times$$

$$\times \left\{ \frac{(\xi+x')^2}{(\xi+x')^2+\xi^2} - \frac{(\xi-x')^2}{(\xi-x')^2+\xi^2} \right\},$$

and

$$J_3(\xi, \xi', \epsilon) = \int_0^\epsilon \frac{dx'}{x'} \frac{(\xi-x')^2}{(\xi-x')^2+\xi^2} \cdot \frac{\partial}{\partial x} \mu(x, \xi')]_{x=\xi-x'} \times$$

$$\times \{\sqrt{(1-(\xi+x')^4)} - \sqrt{(1-(\xi-x')^4)}\}.$$

Taking into account the fact that $\dfrac{x^2}{x^2+\xi^2}$, $\sqrt{(1-x^4)}$ and $\dfrac{\partial}{\partial x} \mu(x, \xi')$ are bounded in $0 \leqslant x \leqslant 1$, $0 \leqslant \xi \leqslant \Lambda$, $0 \leqslant \xi' \leqslant 1$, and also the Lipschitz conditions satisfied by $\sqrt{(1-x^4)}$ and $\dfrac{\partial}{\partial x} \mu(x, \xi')$ in these ranges of x, ξ and ξ' (the latter as given by Corollary 4) it follows that

$$|J_1| \leqslant K_1 \int_0^\epsilon (K_1' |\log x'| + K_1'')\, dx' \leqslant (\bar{K} |\log \epsilon| + \bar{\bar{K}})\epsilon,$$

and

$$|J_3| \leqslant K_2 \int_0^\epsilon \frac{dx'}{x'} (\sqrt{x'} + K'_2 x') \leqslant \bar{K}_2 \sqrt{\epsilon} + \bar{\bar{K}}_2 \epsilon.$$

Now $\dfrac{\partial}{\partial x}\left(\dfrac{x^2}{x^2+\xi^2}\right) = \dfrac{2x}{x^2+\xi^2} - \dfrac{x^2}{(x^2+\xi^2)^2} \cdot 2x = \dfrac{2x\,\xi^2}{(x^2+\xi^2)^2} \leqslant \dfrac{2}{\xi}$.

Hence

$$|J_2| \leqslant K_3 \int_0^\epsilon \frac{dx'}{x'} \cdot \frac{x'}{\xi} = \frac{K_3 \epsilon}{\xi} \leqslant K_3$$

since $\epsilon \leqslant \xi$. It follows that if $\rho(\xi)$ is a continuous monotonic function in $0 \leqslant \xi \leqslant \frac{1}{2}$ and $\rho(0) = 0$, then

$$\left| \frac{\rho(\xi)}{\log \xi} \int\limits_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \right|$$

$$\leqslant \left|\frac{\rho(\Lambda)}{\log \Lambda}\right| \cdot \frac{|I_1 - I_2|}{2} \leqslant \left|\frac{\rho(\Lambda)}{\log \Lambda}\right| \cdot \frac{|J_1| + |J_2| + |J_3| + |I_2|}{2} \to 0$$

as $\Lambda \to 0$, uniformly in $0 < \epsilon \leqslant \xi \leqslant \Lambda$, $0 \leqslant \xi' \leqslant 1$.

CASE C. $0 < \xi < \epsilon$ (and $\xi \leqslant \Lambda$). As before

$$\int\limits_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx$$

$$= \tfrac{1}{2} \int\limits_0^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} \sqrt{(1-x^4)} \frac{\partial}{\partial x} \mu(x, \xi') \frac{dx}{x-\xi} -$$

$$- \tfrac{1}{2} \int\limits_0^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} \sqrt{(1-x^4)} \frac{\partial}{\partial x} \mu(x, \xi') \frac{dx}{x+\xi}$$

$$= \tfrac{1}{2} \int\limits_0^{2\xi} \frac{x^2}{x^2+\xi^2} \frac{\sqrt{(1-x^4)}}{x-\xi} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \; +$$

$$+ \frac{1}{2} \int\limits_{2\xi}^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} \frac{\sqrt{(1-x^4)}}{x-\xi} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \; -$$

$$- \frac{1}{2} \int\limits_0^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} \frac{\sqrt{(1-x^2)}}{x+\xi} \frac{\partial}{\partial x} \mu(x, \xi')\, dx.$$

The first intergal on the right can be seen to be of the same form as $I_1(\xi, \xi', \epsilon)$ defined under Case (B) above, with $\epsilon = \xi$. Hence as shown under Case (B), this integral is bounded in $0 \leqslant \xi < \Lambda$, $0 \leqslant \xi' \leqslant 1$ (and $0 < \epsilon < 1$).

The absolute value of the second integral on the right is

$$\leqslant K \int\limits_{2\xi}^{\xi+\epsilon} \frac{dx}{x-\xi}$$

(since all the other functions under the intergral sign are bounded)

$$= K[\log \epsilon - \log \xi]$$

$$\leqslant 2K\,|\log \xi| \text{ since } 0 < \xi < \epsilon.$$

Similarly the absolute value of the third integral on the right is

$$\leqslant K \int_0^{\xi+\epsilon} \frac{dx}{x+\xi} = K[\log(\epsilon + 2\,\xi) - \log \xi] \leqslant 2K\,|\log \xi|.$$

Hence if $\rho(\xi)$ is continuous and monotonic in (0, 1/2) and $= 0$ for $\xi = 0$, we have

$$\left|\frac{\rho(\xi)}{\log \xi} \int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\,dx\right| \leqslant \frac{\rho(\Lambda)}{\log \xi}\{K_1 + 4K\,|\log \xi|\} \to 0$$

as $\Lambda \to 0$, uniformly in $0 < \xi < \Lambda$, $0 \leqslant \xi' \leqslant 1$, and $\epsilon > \xi$. This proves (ii)$'$.

CASE 3. $\bar{\Lambda} < \xi \leqslant 1$. It can be assumed that $\bar{\Lambda} - \epsilon > 0$. Consider the following possibilities:

Case (B$'$). $\xi + \epsilon \leqslant 1$. As in Case (1)

$$\int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\,dx = \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{f(x, \xi, \xi')}{x - \xi}\,dx,$$

where

$$f(x, \xi, \xi') = \frac{\xi x^2 \sqrt{(1-x^4)}}{(x^2+\xi^2)(x+\xi)} \frac{\partial}{\partial x} \mu(x, \xi').$$

The same arguments as before hold to show that

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{f(x, \xi, \xi')}{x - \xi}\,dx \to 0 \text{ uniformly as } \epsilon \to 0,$$

in $\bar{\Lambda} \leqslant \xi < \xi + \epsilon \leqslant 1, 0 \leqslant \xi' \leqslant 1$.

Case (C$'$). $\xi + \epsilon > 1$. Denoting $1 - \xi$ by $\bar{\epsilon}$ and defining $f(x, \xi, \xi')$ as above, we have

$$\int_{\max(0,\xi-\epsilon)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4 - \xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\,dx$$

$$= \int_{\xi-\epsilon}^{1} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx$$

$$= \int_{\xi-\epsilon}^{\xi-\bar{\epsilon}} \frac{f(x, \xi, \xi')}{x-\xi}\, dx + \frac{1}{2} \int_{\xi-\bar{\epsilon}}^{\xi+\epsilon} \frac{x^2}{x^2+\xi^2} \frac{\sqrt{(1-x^4)}}{x-\xi} \frac{\partial}{\partial x} \mu(x, \xi')\, dx -$$

$$-\frac{1}{2} \int_{\xi-\bar{\epsilon}}^{\xi+\bar{\epsilon}} \frac{x^2}{x^2+\xi^2} \frac{\sqrt{(1-x^4)}}{x+\xi} \frac{\partial}{\partial x} \mu(x, \xi')\, dx..$$

It can be verified that $\dfrac{f(x, \xi, \xi')}{\sqrt{(1-x)}}$ is bounded in the given range of values of x, ξ and ξ'. Also for $\xi-\epsilon \leqslant x \leqslant \xi-\bar{\epsilon}$

$$\left|\frac{\sqrt{(1-x)}}{x-\xi}\right| \leqslant \left|\frac{\sqrt{\{2(\xi-x)\}}}{\xi-x}\right| = \frac{\sqrt{2}}{\sqrt{(\xi-x)}}.$$

Hence

$$\left|\int_{\xi-\epsilon}^{\xi-\bar{\epsilon}} \frac{f(x, \xi, \xi')}{x-\xi}\, dx\right| \leqslant K_1 \int_{\xi-\epsilon}^{\xi-\bar{\epsilon}} \frac{dx}{\sqrt{(\xi-x)}} \leqslant \bar{K}_1 \sqrt{\epsilon}.$$

The second integral on the right can be seen to be of the same form as $I_1(\xi, \xi', \epsilon)$ defined under Case 2 above, with $\epsilon = \bar{\epsilon}$, and which therefore (as shown there) is $= J_1 + J_2 + J_3$, where

$$|J_1| \leqslant (K_2\, |\log \bar{\epsilon}| + \bar{K}_2)\, \bar{\epsilon};$$

$$|J_2| \leqslant K_3 \int_0^{\bar{\epsilon}} \frac{dx'}{x'} \cdot \frac{x'}{\xi} \leqslant \frac{K_3}{\Lambda} \bar{\epsilon};$$

and

$$|J_3| \leqslant K_4 \sqrt{\bar{\epsilon}} + K_5\, \bar{\epsilon}.$$

The absolute value of the third integral on the right is $\leqslant K_6\, \bar{\epsilon}$ since all the functions under the integral sign are bounded in the given range of values of x, ξ and ξ'.

It follows that

$$\int_{\xi-\epsilon}^{1} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \to 0$$

as ϵ and $\bar{\epsilon} \to 0$ independently, or as $\epsilon \to 0$ and $\bar{\Lambda} \to 1$ independently, uniformly in $\bar{\Lambda} < \xi \leqslant 1$, $(\epsilon > 1 - \xi)$, $0 \leqslant \xi' \leqslant 1$. Taking into account the results proved under Case (B′) above and also under Case (1) it follows that

$$\int_{\max(0,\xi-0)}^{\min(\xi+\epsilon,1)} \frac{\xi x^2 \sqrt{(1-x^4)}}{x^4-\xi^4} \frac{\partial}{\partial x} \mu(x, \xi')\, dx \to 0$$

as $\epsilon \to 0$ and $\bar{\Lambda} \to 1$ (independently), uniformly in $\Lambda \leqslant \xi < 1$, $0 \leqslant \xi' \leqslant 1$. This proves (i).

The solving of equation (c) can be reduced as follows to solving, by means of the well-known Fredholm's formula, of a non-homogeneous integral equation of the second kind with a continuous kernal.

Equation (c) can evidently be written as

$$\theta(\xi)\sqrt{(\xi(1-\xi))} = F(\xi) + \int_0^1 \bar{L}(\xi, \xi')\, \theta(\xi')\, d\xi', \qquad (c_2)$$

where

$$F(\xi) = \frac{2\xi\sqrt{(\xi(1-\xi))}}{\sqrt{(1-\xi^4)}} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}}\, dx . \frac{1}{\pi \log 2}$$

$$= \frac{2\,\xi^{3/2}}{\sqrt{\{(1+\xi)(1+\xi^2)\}}} \cdot \frac{1}{\pi \log 2} \int_0^1 \frac{4x}{\sqrt{(1-x^4)}}\, dx$$

and

$$\bar{L}(\xi, \xi') = \frac{2L(\xi, \xi')}{\sqrt{(1-\xi^4)}} \sqrt{(\xi(1-\xi))} = \frac{2L(\xi, \xi')\sqrt{\xi}}{\sqrt{(1+\xi^2)(1+\xi)}}.$$

Now if the equation

$$\alpha(\xi) = F(\xi) + \int_0^1 \frac{\bar{L}(\xi, \xi')\, \alpha(\xi')}{\sqrt{(\xi'(1-\xi'))}}\, d\xi' \qquad (e)$$

has a continuous solution $\alpha(\xi)$, then equation (c) has a solution, namely $\theta(\xi) = \dfrac{\alpha(\xi)}{\sqrt{(\xi(1-\xi))}}$, which satisfies conditions (4) — (7). For, conditions (4), (5), (6) as well as equation (c_2), and hence (c),

are evidently satisfied. (7) follows by dividing both sides of (c) by $\log \xi$ and observing that in view of (ii), Theorem 4,

$$\frac{1}{\log \xi} \cdot \frac{2}{\sqrt{(1-\xi^4)}} \int_0^1 L(\xi, \xi')\, \theta(\xi')\, d\xi'$$

exists and is bounded in the neighbourhood of $\xi = 0$.

Writing $\xi = \sin^2 \beta$, $\xi' = \sin^2\beta'$, $F(\xi) = Y(\beta), \alpha(\xi) = X(\beta)$, $\bar{L}(\xi, \xi') = \frac{1}{2} M(\beta, \beta')$, equation (e) becomes

$$X(\beta) = Y(\beta) + \int_0^{\pi/2} M(\beta, \beta')\, X(\beta')\, d\beta'. \tag{E}$$

Conditions (1), (2) and (3) can now be verified for equation (E). $F(\xi)$ is evidently continuous in (0, 1). Also from Theorem 4 it follows that $\bar{L}(\xi, \xi') = \dfrac{2L(\xi, \xi')\sqrt{\xi}}{\sqrt{\{(1+\xi^2)(1+\xi)\}}}$ is continuous in ξ and ξ' in $0 \leqslant \xi \leqslant 1$, $0 \leqslant \xi' \leqslant 1$. Therefore since $\xi = \sin^2 \beta$ is a continuous function of β in $(0, \pi/2)$, $Y(\beta) = F(\xi)$ and $M(\beta, \beta') = 2L(\xi, \xi')$ are continuous functions in $0 \leqslant \beta \leqslant \pi/2$, and $0 \leqslant \beta \leqslant \pi/2$, $0 \leqslant \beta' \leqslant \pi/2$ respectively. Hence (1) holds.

Since β, the principal value of $\sin^{-1}\sqrt{\xi}$, is a continuous function of ξ in (0, 1), therefore if $X(\beta)$ is a continuous solution of (E) in $(0, \pi/2)$, then $\alpha(\xi) = X(\sin^{-1}\sqrt{\xi})$ is a continuous solution of (e) in (0, 1), and hence $\theta(\xi) = \dfrac{\alpha(\xi)}{\sqrt{(\xi(1-\xi))}}$ is a solution of (c) belonging to (γ). Hence (2) follows from Theorems 3 and 2.

(3) follows by similar arguments, i.e. by showing that the homogeneous equation (c′) has a solution belonging to (γ) provided the corresponding homogeneous equation

$$\bar{\alpha}(\xi) = \int_0^1 \bar{L}(\xi, \xi')\, \bar{\alpha}(\xi')\, \frac{d\xi'}{\sqrt{(\xi'(1-\xi'))}} \tag{e$'$}$$

has a continuous solution, and then transforming (e′) into the form

$$\bar{X}(\beta) = \int_0^{\pi/2} M(\beta, \beta')\,(\bar{X}\beta')\,d\beta'. \qquad \text{(E')}$$

Hence as already pointed out, (E) has a unique solution $x(\beta)$ which can be obtained by Fredholm's formula, and we have

THEOREM 5. *Problem* (A_1) *has a unique continuous solution*

$$\phi(x, y) = \frac{1}{2\pi}\int_0^1 \theta(\xi).\, K(x, y;\, \xi, 0)\, d\xi,$$

where $\theta(\xi) = \dfrac{x(\sin^{-1}\sqrt{\xi})}{\sqrt{(\xi(1-\xi))}}$ *and* $X(\beta)$ *is the (unique) solution of the equation*

$$X(\beta) = Y(\beta) + \int_0^{\pi/2} M(\beta, \beta')\, X(\beta')\, d\beta' \qquad \text{(E)}$$

(M, Y *being defined as above*). *The normal derivative*

$$\left.\frac{\partial\phi(x, y)}{\partial y}\right]_{y=0} = -\,\theta(x),\ 0 < x < 1$$

and satisfies conditions (4) – (7).

A similar method is applicable to the case where there is only one control rod, i.e. the plane segment $y = 0$, $0 \leqslant x \leqslant 1$ is occupied by a control rod of zero thickness. If $\Phi(x, y)$ denotes the density of thermal neutrons, $\phi = \Phi - 1$, and

$$K(x, y;\, \xi, \eta) = \tfrac{1}{2}\pi\, \mathrm{Im}\,\{H_0^1(i\sqrt{[(x-\xi)^2 + (y-\eta)^2]} + H_0^1(i\sqrt{[(x-\xi)^2 + (y+\eta)^2]}\}$$

then

$$2\pi\,\phi(x, y) = \int_0^1 K(x, y;\, \xi, 0)\,\theta(\xi)\, d\xi,$$

where

$$-\,2\pi = \int_0^1 K(x, 0;\, \xi, 0)\,\theta(\xi)\, d\,\xi.$$

The solution of the latter integral equation can be reduced, as before, i.e. by using the results of [3] to the solution of a Fredholm equation of the second kind with a continuous kernal.

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